Curve Sketching Practice

With a partner or two and without the use of a graphing calculator, attempt to sketch the graphs of the following functions. Pertinent aspects of the graph to include (include as many as you can):

- asymptotes (vertical/horizontal)
- domain
- local extrema/regions of increase/decrease
- points of inflection/concavity
- x-intercepts(?)

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1. $f(x) = x^4 - 6x^2$

2. $f(x) = (x^2 - 1)^3$

3. $f(x) = x\sqrt{x^2 + 1}$

4. $f(x) = \dfrac{x}{(x-1)^2}$

Solutions

1

The zeros (x-intercepts) of f:

$$x^4 - 6x^2 = 0 \quad \Rightarrow \quad x^2(x^2 - 6) = 0 \quad \Rightarrow \quad x = 0, \pm\sqrt{6}.$$

The zeros of $f'(x) = 4x^3 - 12x$:

$$4x^3 - 12x = 0 \quad \Rightarrow \quad 4x(x^2 - 3) = 0 \quad \Rightarrow \quad x = 0, \pm\sqrt{3}.$$

f' changes sign at each of these numbers since $f'(-2) < 0$, $f'(-1) > 0$, $f'(1) < 0$, and $f'(2) > 0$. Thus, f has relative minimums $f(-\sqrt{3}) = -9$ and $f(\sqrt{3}) = -9$ and a relative maximum $f(0) = 0$. The zeros of $f''(x) = 12x^2 - 12$:

$$12x^2 - 12 = 0 \quad \Rightarrow \quad 12(x^2 - 1) = 0 \quad \Rightarrow \quad x = -1, 1.$$

f'' changes sign as it passes each of these numbers, since $f''(-2) > 0$, $f''(0) < 0$ and $f''(2) > 0$, so f has points of inflection $(-1, -5)$ (where the graph changes from being concave upward to concave downward) and $(1, -5)$ (concave down to concave up).

2

The zeros of f:

$$(x^2 - 1)^3 = 0 \quad \Rightarrow \quad [(x+1)(x-1)]^3 = 0 \quad \Rightarrow \quad x = -1, 1.$$

The zeros of $f'(x) = 6x(x^2-1)^2$:

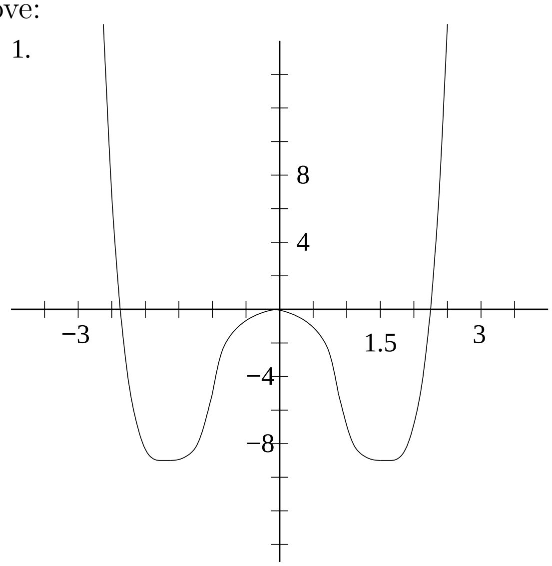
$$6x(x^2-1)^2 \;=\; 0 \qquad \Rightarrow \qquad 6x(x+1)^2(x-1)^2 \;=\; 0 \qquad \Rightarrow \qquad x \;=\; -1,\, 0,\, 1.$$

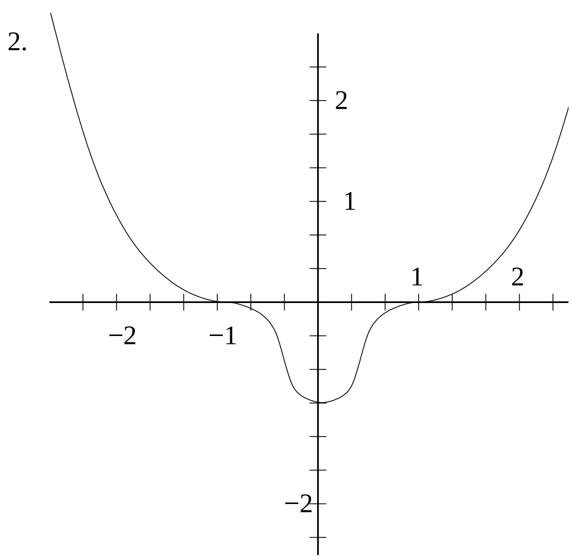
Since $f'(-2) < 0$, $f'(-0.5) < 0$, $f'(0.5) > 0$ and $f'(2) > 0$, f has a relative minimum $f(0) = -1$. The product rule may be used to find f'':

$$f''(x) = 6(x^2-1)^2 + 24x^2(x^2-1) = 6(x^2-1)[(x^2-1)+4x^2] = 6(x^2-1)(5x^2-1).$$

These algebraic simplifications were carried out to factor f'', so that its zeros $x = -1, -1/\sqrt{5}, 1/\sqrt{5}$, and 1 are more easily found. Since $f''(-2) > 0$, $f''(0.5) < 0$, $f''(0) > 0$, $f''(0.5) < 0$ and $f''(-2) > 0$, f has points of inflection at $(-1, 0)$ (where, incidentally, the tangent line is horizontal by the fact that $f'(-1) = 0$ and the graph goes from concave upward to concave downward), at $(-1/\sqrt{5}, -64/125)$, at $(1/\sqrt{5}, -64/125)$ and at $(1, 0)$.

Here are graphs for the functions in problems 1 and 2 sketched using the information we gained above:





3

The zeros of f occur only when the numerator is zero – namely, at $x = 0$. Finding the derivative of f is a matter for the product rule. We have

$$f'(x) = (x^2+1)^{1/2} + x^2(x^2+1)^{-1/2} = \frac{(x^2+1)+x^2}{\sqrt{x^2+1}} = \frac{2x^2+1}{\sqrt{x^2+1}}.$$

Again, f' is zero only when its numerator is zero, and since the equation $2x^2+1 = 0$ has no real solutions, f is not going to have any relative extrema. Turning to the question of concavity we may apply the quotient rule to f' in the form in which it appears above, but I choose here instead to write $f'(x) = (2x^2+1)(x^2+1)^{-1/2}$ and apply the product rule to get f'':

$$\begin{aligned} f''(x) &= 4x(x^2+1)^{-1/2} - x(2x^2+1)(x^2+1)^{-3/2} \\ &= \frac{4x}{\sqrt{x^2+1}} - \frac{2x^3+x}{(x^2+1)\sqrt{x^2+1}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{4x(x^2+1) - 2x^3 - x}{(x^2+1)\sqrt{x^2+1}} \\
&= \frac{2x^3+3x}{(x^2+1)\sqrt{x^2+1}}.
\end{aligned}
$$

As with any fractional expression, f'' may be zero only when its numerator is zero, so we solve

$$2x^2 + 3x = 0 \qquad \Rightarrow \qquad x(2x+3) = 0 \qquad \Rightarrow \qquad x = 0, -\frac{3}{2}.$$

Since $f''(-2) < 0$, $f''(-1) < 0$ and $f''(1) > 0$ we know there is one inflection point at $(0, 0)$, with the concavity of f changing there from downward to upward.

4

f has just one zero at $x = 0$. Writing f in the equivalent form $f(x) = x(x-1)^{-2}$, we get the derivative using the product rule (of course, the quotient rule would also have been an option):

$$f'(x) = (x-1)^{-2} - 2x(x-1)^{-3} = \frac{1}{(x-1)^2} - \frac{2x}{(x-1)^3} = \frac{(x-1)-2x}{(x-1)^3} = \frac{-x-1}{(x-1)^3}.$$

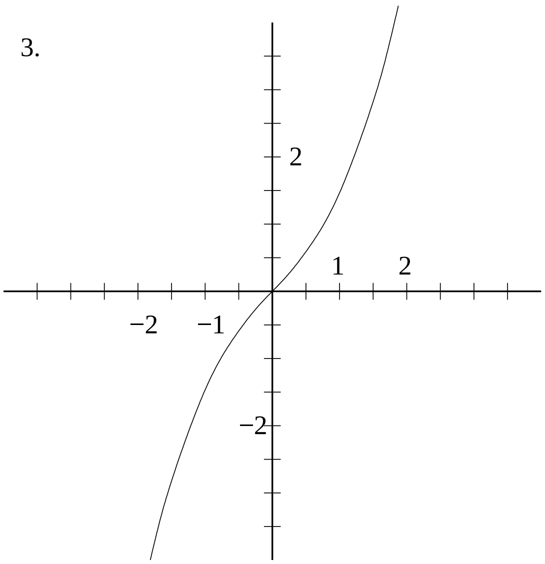
This shows that f' has only one zero at $x = -1$. Like usual we should check the sign of f' on both sides of this number. What is different about this particular example is that if we check the sign of, say, $f'(0)$, we may **not** presume that this is the sign of f' for **all** numbers $x > -1$, rather this is true just for those numbers $-1 < x < 1$. $x = 1$ is a number not actually in the domain of f — actually the site of a vertical asymptote for this function — and we have to check the sign of f' for numbers $x > 1$ separately. Since $f'(-2) < 0$, $f'(0) > 0$ and $f'(2) < 0$ we have the f decreases on $-\infty < x < -1$, reaching a local minimum $f(-1) = -\frac{1}{4}$, and increases on $-1 < x < 1$ (with the values of f approaching $+\infty$), and decreasing (coming down from $+\infty$) for $1 < x < \infty$. We now glean what we can from the second derivative. Writing $f'(x)$ in the equivalent form $(-x-1)(x-1)^{-3}$, we get

$$
\begin{aligned}
f''(x) &= -(x-1)^{-3} - 3(-x-1)(x-1)^{-4} \\
&= \frac{-1}{(x-1)^3} + \frac{3x+3}{(x-1)^4} \\
&= \frac{-(x-1)+3x+3}{(x-1)^4} \\
&= \frac{2x+4}{(x-1)^4}.
\end{aligned}
$$

We determine possible points of inflection first by determining where the numerator of f'' is zero. The only solution to $2x + 4 = 0$ is $x = -2$. We note that $f''(-3) < 0$, $f''(0) > 0$, $f''(2) > 0$ (we check numbers from the two intervals $-2 < x < 1$ and $1 < x < +\infty$ separately because of the break in the domain at $x = 1$). Thus f is concave down on the interval $-\infty < x < -2$, has a point of inflection at $(-2, -2/9)$, and is concave up on each of the intervals $-2 < x < 1$ and $1 < x < +\infty$.

Here are graphs sketched for problems 3 and 4:

3.



4.