MATH 162: Calculus II

Framework for Wed., Feb. 28

Limits and Continuity

Today's Goal: To understand the meaning of limits and continuity of functions of 2 and 3 variables.

Geometry of the Domain Space

Definition: (Distance in $\mathbb{R}^3$): Suppose that (x_1, y_1, z_1) and (x_2, y_2, z_2) are points in $\mathbb{R}^3$. The *distance* between these two points is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

If our two points have the same z-value (for instance, if they both lie in the xy-plane), then the distance between them is just

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

We employ these definitions for distance in $\mathbb{R}^2$, $\mathbb{R}^3$ to define circles, disks, spheres and balls.

Definition: A *circle* in $\mathbb{R}^2$ centered at (a, b) with radius r is the set of points (x, y) satisfying

$$(x - a)^2 + (y - b)^2 \ = \ r^2.$$

Given such a circle C, the set of all points on or inside C is a *closed disk*. The set of points inside but not on C is an *open disk*.

Definition: A *sphere* in $\mathbb{R}^3$ centered at (a, b, c) with radius r is the set of points (x, y, z) satisfying

$$(x - a)^2 + (y - b)^2 + (z - c)^2 \ = \ r^2.$$

The inside of a sphere is called a *ball*, and can be closed or open depending on whether the (whole) sphere is included.

Open intervals in $\mathbb{R}$ are sets of the form $a < x < b$ (also written using interval notation (a, b)), where neither endpoint a, b is included in the set. One observation about such sets is that, if you take any $x \in (a, b)$, there is a value of $r > 0$, perhaps quite small, for which the interval $(x - r, x + r)$ is wholly contained inside (a, b). We build on that idea when defining various kinds of subsets of $\mathbb{R}^2$.

Definition: Let R be a region of the xy-plane and (x_0, y_0) a point (perhaps in R, perhaps not). We call (x_0, y_0) an *interior point* of R if there is an open disk of positive radius centered at (x_0, y_0) such that every point in this disk lies inside R.

We call (x_0, y_0) a *boundary point* of R if every disk with positive radius centered at (x_0, y_0) contains both a point that is in R and a point that isn't in R.

The region R is said to be *open* if all points in R are interior points of R.

The region R is said to be *closed* if all boundary points of R are in R.

The region R is said to be *bounded* if it lies entirely inside a disk of finite radius.

Examples:

1. An open disk is open. A closed disk is closed. For both, the boundary points are those found on the enclosing circle.
2. The upper half-plane R consisting of points (x, y) for which $y > 0$ is an open, unbounded set. The boundary points of R are precisely those points found along the x-axis, none of which are contained in R.
3. If $y = f(x)$ is a continuous function on the interval $a \leq x \leq b$, then the graph of f is a closed, bounded set made up entirely of boundary points.
4. The set of points (x, y) satisfying $x \geq 0$, $y \geq 0$ and $y < x + 1$ is bounded, but neither open nor closed.
5. The only nonempty region of the plane which is both open and closed is the entire plane $\mathbb{R}^2$.

Limits and Continuity

The idea behind the statement

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) \;=\; L$$

for a function f of two variables is that you can make the value of $f(x, y)$ as close to L as you like by focusing only on (x, y) in the domain of f which are inside an open disk centered at (x_0, y_0) of some positive radius. The official definition follows.

Definition: The *limit* of f as (x, y) approaches (x_0, y_0) is L if for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that, for all (x, y) in dom(f),

$$\text{if} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \epsilon.$$

Remarks:

1. All the limit laws of functions of a single variable—those stated in Section 2.2—have analogs for functions of 2 variables. For instance, the analog to Rule 5 on p. 65 goes like this:

> **Theorem**: Suppose
> $$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$
> (That is, suppose both these limits exist, and call them L, M respectively.) If $M \neq 0$, then
> $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}.$$

2. We recall that, for a function f of a single variable and a point x_0 interior to the domain of f, the statement $\lim_{x\to x_0} f(x) = L$ requires the values of f to approach L as the x-values approach x_0 from both the left and the right. The requirement at interior points (x_0, y_0) to the domain of a function f of 2 variables is even more strict. Any path of (x, y)-values within the domain of f traversed en route to (x_0, y_0) should produce function values which approach L.

   Example: The limits $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+y^2}$ and $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^4}$ do not exist.

3. One defines continuity for functions of two variables in an identical fashion as for functions of a single variable.

> **Definition**: Suppose (x_0, y_0) is in the domain of f. We say that f *is continuous at* (x_0, y_0) if
> $$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$$
> (i.e., if this limit exists and has the same value as $f(x_0, y_0)$). We say f *is continuous* (or f *is continuous throughout its domain*) if f is continuous at each point in its domain.

   Example: The functions $f(x,y) := \dfrac{4xy}{x^2+y^2}$ and $g(x,y) := \dfrac{4xy^2}{x^2+y^4}$ are continuous at all points (x, y) except $(0, 0)$.

4. The notions of *distance*, *open* and *closed balls*, *boundary point*, *interior point*, *open* and *closed sets*, *limits* and *continuity* may all be generalized to functions of n variables, where n is any integer greater than 1.