Bessel's Equation

The family of differential equations known as **Bessel equations of order** $p \geq 0$ look like:

$$x^2 y'' + xy' + (x^2 - p^2)y = 0, \qquad x > 0. \tag{1}$$

Remarks:

- This is a differential equation (MATH 231). Goal: To find a function $y(x)$ which solves (i.e., satisfies Equation (1). I give this example as an application of power series; it is unlikely to be repeated by a MATH 231 instructor.
- DEs are mathematical models: $dv/dt = g$, $F = m(d^2s/dt^2)$ (Newton's 2nd Law), etc. Friedrich Wilhelm Bessel (1784–1846) studied this type of DE in conjunction with considering disturbances in planetary motion.
- Along with the original context in which they arose, these DEs come up when solving PDEs involving the Laplacian in polar/cylindrical coordinates (such as in understanding the vibrations of a circular drum).

We consider the Bessel equation of order 0:

$$x^2 y'' + xy' + x^2 y = 0, \qquad x > 0. \tag{2}$$

We assume that this equation has a power series solution centered at 0—that is, that the solution y has the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n,$$

so that

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad \text{and} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

Substituting these expressions into Equation (2), we get

$$\begin{aligned}
0 &= x^2 y'' + xy' + x^2 y = x^2 \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x \sum_{n=1}^{\infty} n c_n x^{n-1} + x^2 \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} \\
&= \sum_{n=2}^{\infty} n(n-1)c_n x^n + \left(c_1 x + \sum_{n=2}^{\infty} n c_n x^n \right) + \sum_{j=2}^{\infty} c_{j-2}\, x^j \qquad \text{(substituting } j = n+2 \text{ in final summation)} \\
&= c_1 x + \sum_{n=2}^{\infty} [n(n-1)c_n + n c_n + c_{n-2}]\, x^n = c_1 x + \sum_{n=2}^{\infty} (n^2 c_n + c_{n-2}) x^n \\
&= c_1 x + (4c_2 - c_0)x^2 + (9c_3 + c_1)x^3 + (16c_4 + c_2)x^4 + \cdots + (n^2 c_n + c_{n-2})x^n + \cdots .
\end{aligned}$$

Now, we equate coefficients of various powers of x, noting that the left side of the equation, being zero, has zero x-terms, zero x^2-terms, zero x^3-terms, etc.

$$
\begin{array}{rlcl}
x^0: & 0 = 0, & & \\
x^1: & c_1 = 0, & & \\
x^2: & 2^2c_2 + c_0 = 0 & \Rightarrow & c_2 = \dfrac{-1}{2^2}\,c_0, \\
x^3: & 3^2c_3 + c_1 = 0 & \Rightarrow & c_3 = \dfrac{-1}{3^2}\,c_1 = 0, \\
x^4: & 4^2c_4 + c_2 = 0 & \Rightarrow & c_4 = \dfrac{-1}{2^2(2)^2}c_2 = \left[\dfrac{-1}{2^2(2)^2}\right]\left[\dfrac{-1}{2^2}\,c_0\right] = \dfrac{(-1)^2}{2^4(2!)^2}\,c_0, \\
x^5: & 5^2c_5 + c_3 = 0 & \Rightarrow & c_5 = \dfrac{-1}{5^2}\,c_3 = 0, \\
x^6: & 6^2c_6 + c_4 = 0 & \Rightarrow & c_6 = \dfrac{-1}{2^2\cdot 3^2}c_4 = \left[\dfrac{-1}{2^2\cdot 3^2}\right]\left[\dfrac{(-1)^2}{2^4(2!)^2}\,c_0\right] = \dfrac{(-1)^3}{2^6(3!)^2}\,c_0, \\
\vdots & & & \\
x^{2k-1}: & (2k-1)^2c_{2k-1} + c_{2k-3} = 0 & \Rightarrow & c_{2k-1} = \dfrac{-1}{(2k-1)^2}\,c_{2k-3} = 0, \\
x^{2k}: & (2k)^2c_{2k} + c_{2k-2} = 0 & \Rightarrow & c_{2k} = \dfrac{-1}{2^2k^2}\,c_{2k-2} = \dfrac{(-1)^k}{2^{2k}(k!)^2}\,c_0,
\end{array}
$$

and so on. Taking $c_0 = K$, we have

$$
\begin{aligned}
y(x) &= K + 0x + \frac{-1}{2^2}\,Kx^2 + 0x^3 + \frac{(-1)^2}{2^4(2!)^2}\,Kx^4 + 0x^5 + \frac{(-1)^3}{2^6(3!)^2}\,Kx^6 + \cdots \\
&= K\left[1 + \frac{-1}{2^2(1!)^2}\,x^2 + \frac{(-1)^2}{2^4(2!)^2}\,x^4 + \frac{(-1)^3}{2^6(3!)^2}\,x^6 + \cdots\right] = K\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}\,x^{2n}.
\end{aligned}
$$

The power series

$$J_0(x) := \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}\,x^{2n},$$

is called the **Bessel function of the first kind of order 0**[1].

The method demonstrated above, in which a power series solution is sought for a DE, is commonly used for *linear, non-constant coefficient ordinary DEs*. A nice exercise (entirely optional) is to adapt this procedure to the solution of Airy's (differential) equation

$$y'' - xy = 0.$$

It should give rise to two separate functions[2]

$$y_1(x) = 1 + \sum_{n=1}^{\infty}\frac{x^{3n}}{(2\cdot 3)(5\cdot 6)\cdots[(3n-1)\cdot(3n)]}, \qquad \text{and} \qquad y_2(x) = x + \sum_{n=1}^{\infty}\frac{x^{3n+1}}{(3\cdot 4)(6\cdot 7)\cdots[(3n)\cdot(3n+1)]}.$$

[1]There is also a **Bessel function of the second kind** of order 0. As far as I know, **Close Encounters of the Third Kind** is still fictional.

[2]See the website http://www.sosmath.com/diffeq/series/series04/series04.html for details.