Cramer's Rule

Cramer's rule provides a method for solving a system of linear algebraic equations for which the associated matrix problem $\mathbf{Ax} = \mathbf{b}$ has a coefficient matrix which is *nonsingular*. It is of no use if this criterion is not met and, considering the effectiveness of algorithms we have learned already for solving such a system (inversion of the matrix $\mathbf{A}$, and Gaussian elimination, specifically), it is not clear why we need yet another method. Nevertheless, it is a tool (some) people use, and should be recognized/understood by you when you run across it. We will describe the method, but not explain why it works, as this would require a better understanding of determinants than our time affords.

So, let us assume the n-by-n matrix $\mathbf{A}$ is nonsingular, that $\mathbf{b}$ is a known vector in $\mathbb{R}^n$, and that we wish to solve the equation $\mathbf{Ax} = \mathbf{b}$ for an unknown (unique) vector $\mathbf{x} \in \mathbb{R}^n$. Cramer's rule requires the construction of matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$, where each $\mathbf{A}_j$, $1 \leqslant j \leqslant n$ is built from the original $\mathbf{A}$ and $\mathbf{b}$. These are constructed as follows: the j^{th} column of $\mathbf{A}$ is replaced by $\mathbf{b}$ to form $\mathbf{A}_j$.

Example 1: Construction of $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ when $\mathbf{A}$ is 3-by-3

Suppose $\mathbf{A} = (a_{ij})$ is a 3-by-3 matrix, and $\mathbf{b} = (b_i)$, then

$$\mathbf{A}_1 = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}, \quad \text{and} \quad \mathbf{A}_3 = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}.$$

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Armed with these $\mathbf{A}_j$, $1 \leqslant j \leqslant n$, the solution vector $\mathbf{x} = (x_1, \ldots, x_n)$ has its j^{th} component given by

$$x_j = \frac{|\mathbf{A}_j|}{|\mathbf{A}|}, \qquad j = 1, 2, \ldots, n. \tag{1}$$

It should be clear from this formula why it is necessary that $\mathbf{A}$ be nonsingular.

Example 2:

Use Cramer's rule to solve the system of equations

$$\begin{aligned} x + 3y + z - w &= -9 \\ 2x + y - 3z + 2w &= 51 \\ x + 4y + 2w &= 31 \\ -x + y + z - 3w &= -43 \end{aligned}$$

Here, **A** and **b** are given by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 1 & -1 \\ 2 & 1 & -3 & 2 \\ 1 & 4 & 0 & 2 \\ -1 & 1 & 1 & -3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -9 \\ 51 \\ 31 \\ -43 \end{pmatrix}, \qquad \text{so} \qquad |\mathbf{A}| = \begin{vmatrix} 1 & 3 & 1 & -1 \\ 2 & 1 & -3 & 2 \\ 1 & 4 & 0 & 2 \\ -1 & 1 & 1 & -3 \end{vmatrix} = -46.$$

Thus,

$$x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} -9 & 3 & 1 & -1 \\ 51 & 1 & -3 & 2 \\ 31 & 4 & 0 & 2 \\ -43 & 1 & 1 & -3 \end{vmatrix} = \frac{-230}{-46} = 5,$$

$$y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 1 & -9 & 1 & -1 \\ 2 & 51 & -3 & 2 \\ 1 & 31 & 0 & 2 \\ -1 & -43 & 1 & -3 \end{vmatrix} = \frac{-46}{-46} = 1,$$

$$z = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 1 & 3 & -9 & -1 \\ 2 & 1 & 51 & 2 \\ 1 & 4 & 31 & 2 \\ -1 & 1 & -43 & -3 \end{vmatrix} = \frac{276}{-46} = -6,$$

$$w = \frac{|\mathbf{A}_4|}{|\mathbf{A}|} = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 1 & 3 & 1 & -9 \\ 2 & 1 & -3 & 51 \\ 1 & 4 & 0 & 31 \\ -1 & 1 & 1 & -43 \end{vmatrix} = \frac{-506}{-46} = 11,$$

yielding the solution $\mathbf{x} = (x, y, z, w) = (5, 1, -6, 11)$.

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