Gaussian Elimination

In what follows we focus on solving systems of m **linear** equations in n unknown variables $x_1, \ldots,$ x_n. In order for the system to be *linear*, each time a variable x_j appears in some term, none of the others appear in that same term, and x_j appears to the first power. Essentially, this means if any x_j is in an equation, its appearance (when simplified) is in a term like ax_j; i.e., rescaled by a constant.

We set forward examples and solve them using the standard method discussed in high school algebra courses: elimination.

Example 1:

Consider the system of equations: $\begin{cases} x - 2y &= 1 \\ 3x + 2y &= 11 \end{cases}$

As equations:

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned}$$

A matrix storing just the coefficients:

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & 11 \end{bmatrix}$$

Replacing the 2[nd] equation: $R_2 - 3R_1 \to R_2$:

$$\begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 8 & 8 \end{bmatrix}$$

The first variable (x) remains in the first equation, but has been eliminated from the 2[nd] equation. Note that we could achieve this working exclusively on the matrix of coefficients (right side)—subtracting three multiples of the first row from the row. The resulting matrix is said to be in **row echelon form**. (Follow the link https://en.wikipedia.org/wiki/Row_echelon_form to learn what this term means.)

Pause to think through how, using only this last matrix, you would determine the values of the variables x and y.

We could carry out a few more algebraic steps:

Divide 2[nd] equation by 8: $(1/8)R_2 \to R_2$

$$\begin{aligned} x - 2y &= 1 \\ y &= 1 \end{aligned}$$

Associated matrix:

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Replacing the 1[st] equation: $R_1 + 2R_1 \to R_1$:

$$\begin{aligned} x &= 3 \\ y &= 1 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

From the resulting matrix, said to be in **reduced row echelon form**, it is even easier to pick off the values of x and y. (The Wikipedia link explains this term, too.) Having begun with the equations of two lines in the plane, we can now say those lines intersect at the unique point

$(x, y) = (3, 1)$.

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It is crucial that you stop and affix in your mind the meanings of **row echelon form** (often called simply **echelon form**) and **reduced row echelon form** (or RREF), as well as learn to identify when a given matrix form has such a form. When you have read the Wikipedia descriptions (linked above) and feel you are ready, try to identify which of the following matrices have echelon form, RREF (which is just a special type of echelon form), or neither. You can learn whether your answers are correct when you do the **Socrative** quiz at the end.

(a) $\begin{bmatrix} 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 0 & 4 & 1 \\ 0 & 5 & 0 & 2 \\ -2 & 1 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

We will work toward solving systems of linear equations using *only* matrices. But let's work another example with equations and matrices side-by-side, one in which an oddity occurs.

Example 2:

Consider the system of equations: $\left\{ \begin{array}{rcr} 3x - 6y &=& 11 \\ 2x - 4y &=& 8 \end{array} \right.$

As equations

$$\begin{array}{rcr} 3x - 6y &=& 11 \\ 2x - 4y &=& 8 \end{array}$$

The corresponding matrix:

$$\begin{bmatrix} 3 & -6 & 11 \\ 2 & -4 & 8 \end{bmatrix}$$

Divide the 1$^{\text{st}}$ equation by 3: $(1/3)R_1 \to R_1$:

$$\begin{array}{rcr} x - 2y &=& 11/3 \\ 2x - 4y &=& 8 \end{array} \qquad \begin{bmatrix} 1 & -2 & 11/3 \\ 2 & -4 & 8 \end{bmatrix}$$

Now $R_2 - 2R_1 \to R_2$:

$$\begin{array}{rcr} x - 2y &=& 11/3 \\ 0 &=& 2/3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 11/3 \\ 0 & 0 & 2/3 \end{bmatrix}$$

Down the left column, our elimination steps resulted in the removal of *both* x and y. Or focusing on how these same operations have affected the original matrix, we see row 2 has zeros in the columns corresponding to x and y. The final entry, the one that corresponds to the right side of the equation, is not zero, however, giving us the nonsense statement $0 = 2/3$.

Summarizing the matrix (right column) version of things, we began with a matrix that corresponded to our system, reduced it to row echelon form, and the final version offers us a row with the untenable statement $0 = 2/3$. We conclude the system has no solution. When you consider the original equations, it's not difficult to see they expressed two parallel lines, so there is no point of intersection to be found.

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Example 3:

Let's look at a system with 3 variables: $\left\{\begin{array}{rcr} 3x_1 - x_2 + x_3 &=& -2 \\ -x_1 + x_2 + x_3 &=& 4 \\ 5x_1 + 2x_2 + 9x_3 &=& 15 \end{array}\right.$

This time I will form the corresponding matrix, and work exclusively with it, until (row) echelon form is reached.

$$\begin{bmatrix} 3 & -1 & 1 & -2 \\ -1 & 1 & 1 & 4 \\ 5 & 2 & 9 & 15 \end{bmatrix} \qquad R_1 \leftrightarrow R_2 \qquad \begin{bmatrix} -1 & 1 & 1 & 4 \\ 3 & -1 & 1 & -2 \\ 5 & 2 & 9 & 15 \end{bmatrix}$$

$$(-1)R_1 \to R_1 \qquad \begin{bmatrix} 1 & -1 & -1 & -4 \\ 3 & -1 & 1 & -2 \\ 5 & 2 & 9 & 15 \end{bmatrix}$$

$$(-3)R_1 + R_2 \to R_2 \qquad \begin{bmatrix} 1 & -1 & -1 & -4 \\ 0 & 2 & 4 & 10 \\ 5 & 2 & 9 & 15 \end{bmatrix}$$

$$(-5)R_1 + R_3 \to R_3 \qquad \begin{bmatrix} 1 & -1 & -1 & -4 \\ 0 & 2 & 4 & 10 \\ 0 & 7 & 14 & 35 \end{bmatrix}$$

$$(1/2)R_2 \to R_2 \qquad \begin{bmatrix} 1 & -1 & -1 & -4 \\ 0 & 1 & 2 & 5 \\ 0 & 7 & 14 & 35 \end{bmatrix}$$

$$R_3 - 7R_2 \to R_3 \qquad \begin{bmatrix} 1 & -1 & -1 & -4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have arrived at (row) echelon form and, with one more step, namely adding row 2 to row

1, we arrive at RREF (reduced row echelon form):

$$R_1 + R_2 \to R_1 \qquad \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Translating back to equations, this RREF says

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 + 2x_3 &= 5 \\ 0 &= 0 \end{aligned}$$

The last equation is *vacuous* (doesn't add any information), but it is not false. A general principle is that variables provide freedoms while equations act as constraints, taking freedoms away. Applying this principle to the example at hand would predispose us to guess that the three freedoms were balanced by three constraints, resulting in precisely one solution point (x_1, x_2, x_3) (or even that there might be *no solution*, as in Example 2, when the two variables were subjected to two incompatable constraints). While there are many linear systems of 3 equations in 3 unknowns that go that way, the case here is that echelon form has revealed *these* three equations actually place only *two* constraints on the variables (two meaningful ones, along with the uninformative statement $0 = 0$). So, we have the situation that one variable can have its values chosen freely (i.e., one degree of freedom remains), and once it is chosen, the values of the other variables are fixed.

It is convenient to solve the above (informative equations) for x_1 and x_2 respectively:

$$\left.\begin{aligned} x_1 + x_3 &= 1 \\ x_2 + 2x_3 &= 5 \end{aligned}\right\} \qquad \Rightarrow \qquad \left\{\begin{aligned} x_1 &= 1 - x_3 \\ x_2 &= 5 - 2x_3 \end{aligned}\right.$$

and think of x_3 as a **free variable**. Our original system of equations has infinitely many solutions, one for each choice of $x_3 \in \mathbb{R}$. We will generally express these solutions in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - x_3 \\ 5 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \qquad x_3 \in \mathbb{R}.$$

Note, finally, that had we used the notation of Calculus (first introduced in MATH 172, and used further in MATH 271), our answer would have been

$$\langle x_1, x_2, x_3 \rangle = \langle 1, 5, 0 \rangle + t \langle -1, -2, 1 \rangle, \qquad t \in \mathbb{R},$$

which, depending on your retention of Calculus concepts, you might recognize as expressing (parametrically) a *line* in space. If you also recall that equations such as

$$ax + by + cz = d$$

in variables x, y, and z have graphs which are *planes*, then the geometric interpretation of this problem is that we were looking for points of intersection between three planes

$$3x_1 - x_2 + x_3 = -2, \qquad -x_1 + x_2 + x_3 = 4, \qquad \text{and} \qquad 5x_2 + 2x_2 + 9x_3 = 15,$$

and learned that these planes intersect not in just a single point, but along an entire line, as if they were separate pages in a book with a common spine.

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The subject of this handout is **Gaussian elimination**, which is what we call it when we work with the matrix of a linear system of equations and take it to row echelon form (or even further, to reduced row echelon form). I hope it becomes obvious that, once we have echelon form, we can easily separate those systems that have *no solution* from those that do, and further distinguish solvable (or **consistent**) systems that have precisely one solution from those that have infinitely many solutions (i.e, where at least one degree of freedom remains). The real goal of Gaussian elimination is, then, twofold:

- to distinguish between systems of linear equations having no solution, precisely one solution, and infinitely many solutions, and
- for those systems that have solutions, identifying them, as we did in Example 1 where there was the lone solution $\langle 3, 1\rangle$, and in Example 3 where there were infinitely many $\langle x_1, x_2, x_3\rangle = \langle 1, 5, 0\rangle + t\,\langle -1, -2, 1\rangle$, $t \in \mathbb{R}$.

It is the process of morphing from the linear system's original matrix to an echelon form which comprises the real work of Gaussian elimination. There are just three types of steps/transformations which are legal. These three "moves" are collectively known as **elementary row operations** (EROs), and they are:

- **swapping two rows**, sometimes denoted by $R_i \leftrightarrow R_j$. You can see how, if a system led to the matrix on the left, a row swap would be just the thing to quickly arrive at row echelon form.
$$\begin{bmatrix} 0 & 2 & -1 & 3 \\ 1 & 0 & 5 & 0 \end{bmatrix} \qquad R_1 \leftrightarrow R_2 \qquad \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 2 & -1 & 3 \end{bmatrix}.$$
- **multiplying a row by a nonzero constant**, sometimes denoted by $cR_i \to R_i$. The first step of Example 2 was an instance of this ERO:
$$\begin{bmatrix} 3 & -6 & 11 \\ 2 & -4 & 8 \end{bmatrix} \qquad (1/3)R_1 \to R_1 \qquad \begin{bmatrix} 1 & -2 & 11/3 \\ 2 & -4 & 8 \end{bmatrix}.$$
- **adding a multiple of one row to another row**, sometimes denoted by $cR_i + R_j \to R_j$. We have done this in several examples, with one instance being the second step of Example 2:
$$\begin{bmatrix} 1 & -2 & 11 \\ 2 & -4 & 8 \end{bmatrix} \qquad (-2)R_1 + R_2 \to R_2 \qquad \begin{bmatrix} 1 & -2 & 11/3 \\ 0 & 0 & 2/3 \end{bmatrix}.$$

If you own a TI-8x calculator and think you would like to take advantage of its functionality in carrying our EROs, I suggest you watch this video https://www.youtube.com/watch?v=ePtrvmMUMXU. The man doing the demonstration shows you both how to take a matrix directly to RREF *in one step*, as well how to carry out using individual EROs. While the RREF function is very useful, I expect you to be able to do the individual EROs—both to be able to map out a sequence of EROs that lead to echelon form (or to RREF), and to carry them out successfully.

Another good video using only the calculator's RREF function, but doing so on at least one problem with a free variable, is this one https://www.youtube.com/watch?v=_4aKp_ZbTEI.

Work through the examples in this handout and the videos to the point that you feel you have a good handle on the material. When you are ready, proceed to take the quiz.

Socrative quiz

Some people will have installed a Socrative Student app on their phones. If you haven't used Socrative before and do not have the app, the easiest way to access the quiz is to point a web browser at the website https://b.socrative.com/login/student/, enter as Room Name **SCOFIELD3894**, hit "Join", and take the quiz called "MATH 231: GE-Intro". You will need to be able to view the matrices (a)–(f) in the box at the top of the 2^{nd} page of this handout while taking the quiz.