MATH 231
Laplace transform shift theorems

There are **two** results/theorems establishing connections between shifts and exponential factors of a function and its Laplace transform.

Theorem 1: If $f(t)$ is a function whose Laplace transform $\mathcal{L}[f(t)](s) = F(s)$, then

A. $\mathcal{L}\left[e^{at} f(t)\right](s) \; = \; F(s-a)$, and

B. $\mathcal{L}[H(t-a)\, f(t-a)](s) \; = \; e^{-as} F(s)$.

Neither of these theorems is strictly necessary for computing Laplace transforms—i.e., when going from the time domain function $f(t)$ to its frequency domain counterpart $\mathcal{L}[f(t)](s)$. Such transforms can be computed directly from the definition of Laplace transform $\mathcal{L}[f(t)](s) = \int_0^\infty e^{-st} f(t)\, dt$.

Example 1:

We compute

(a) $\mathcal{L}\left[te^{2t}\right](s)$, and

(b) $\mathcal{L}\left[H(t-3)\, e^{t-3}\right]$

directly from the definition.

For part (a),

$$\begin{aligned}\mathcal{L}\left[te^{2t}\right](s) &= \int_0^\infty e^{-st} te^{2t}\, dt = \int_0^\infty te^{-(s-2)t}\, dt = \int_0^\infty te^{-st}\, dt\Big|_{s\mapsto s-2} = \mathcal{L}[t](s-2) \\ &= \frac{1}{s^2}\Big|_{s\mapsto s-2} = \frac{1}{(s-2)^2}.\end{aligned}$$

For part (b),

$$\begin{aligned}\mathcal{L}\left[H(t-3)\, e^{t-3}\right] &= \int_0^\infty e^{-st} H(t-3)\, e^{t-3}\, dt = \int_3^\infty e^{-st} e^{t-3}\, dt \\ &= \int_0^\infty e^{-s(u+3)} e^u\, du \qquad \text{(by substitution: } u = t-3) \\ &= e^{-3s} \int_0^\infty e^{-su} e^u\, du = e^{-3s} \int_0^\infty e^{-st} e^t\, dt \quad \text{(the \textit{name} of the variable of integration is immaterial)} \\ &= e^{-3s}\, \mathcal{L}\left[e^t\right] = e^{-3s} \frac{1}{s-1}.\end{aligned}$$

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Using shift theorems for inverse Laplace transforms

It is in finding *inverse* Laplace transforms where Theorems A and B are indispensible.

Example 2:

Find the inverse Laplace transform for each of the functions

(a) $\dfrac{se^{-2s}}{s^2+9}$ (b) $\dfrac{3}{(s+1)^3}$ (c) $\dfrac{2s}{s^2-4s+5}$

Our function in part (a) has an exponential factor, much like in Theorem B. Here,

$$e^{-2s}\frac{s}{s^2+9} = e^{-2s}F(s), \qquad \text{where} \qquad F(s) = \frac{s}{s^2+9} = \mathcal{L}\left[\cos(3t)\right](s).$$

Thus,

$$\mathcal{L}^{-1}\left[e^{-2s}\frac{s}{s^2+9}\right](t) \;=\; H(t-2)\cos(3(t-2)).$$

The function in part (b) does not look like an entry in the Laplace transform table I provide: http://www.calvin.edu/~scofield/courses/m231/F14/table_of_Laplace_transforms.pdf
It is, in fact, a modified version of the table entry $n!/s^{n+1}$ with $n=2$ but shifted left 1 unit, i.e.,

$$\frac{3}{(s+1)^3} \;=\; \frac{3}{s^3}\bigg|_{s\mapsto s+1} \;=\; \frac{3}{2}\cdot\frac{2!}{s^3}\bigg|_{s\mapsto s-(-1)}.$$

Since

$$\mathcal{L}^{-1}\left[\frac{3}{2}\cdot\frac{2!}{s^3}\right](t) \;=\; \frac{3}{2}\mathcal{L}^{-1}\left[\cdot\frac{2!}{s^3}\right](t) \;=\; \frac{3}{2}t^2,$$

it follows from Theorem A that

$$\mathcal{L}^{-1}\left[\frac{3}{(s+1)^3}\right](t) \;=\; \mathcal{L}^{-1}\left[\frac{3}{2}\cdot\frac{2!}{s^3}\bigg|_{s\mapsto s-(-1)}\right](t) \;=\; \frac{3}{2}t^2e^{-t}.$$

The function in part (c) also does not look like an entry in the table of Laplace transforms found at the link above. The denominator is, in fact, an **irreducible quadratic (over the reals)**, having no real roots. But a quadratic has a parabolic graph, and any parabola may be obtained from the graph of $y=x^2$ via a sequence of shifts and stretches. We can find the shift involved through completing the square:

$$s^2-4s+5 \;=\; s^2-4s+4+1 \;=\; (s-2)^2+1,$$

which means the graph of s^2-4s+5 is the same as the graph of s^2+1 but shifted 2 units to the right. To use Theorem A, we need *all* instances of s to be similarly shifted, so we write

$$\frac{2s}{s^2-4s+5} \;=\; \frac{2s}{(s-2)^2+1} \;=\; \frac{2(s-2+2)}{(s-2)^2+1} \;=\; \frac{2(s-2)+4}{(s-2)^2+1} \;=\; \frac{2s+4}{s^2+1}\bigg|_{s\mapsto s-2}.$$

[Take a moment to plot, together, the functions $2x/(x^2-4x+5)$ and $(2x+4)/(x^2+1)$. Observe that the graph of the former is identical to that of the latter, except shifted right 2 units.] Since

$$\mathcal{L}^{-1}\left[\frac{2s+4}{s^2+1}\right] \;=\; 2\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]+4\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] \;=\; 2\cos t+4\sin t,$$

it follows from Theorem A that

$$\mathcal{L}^{-1}\left[\frac{2s}{s^2-4s+5}\right] \;=\; \mathcal{L}^{-1}\left[\frac{2s+4}{s^2+1}\bigg|_{s\mapsto s-2}\right] \;=\; e^{2t}\left(2\cos t+4\sin t\right).$$

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In some cases, we employ partial fraction expansion as part of finding the inverse Laplace transform.

Example 3:

Find the inverse Laplace transform for each of the functions

(a) $\dfrac{8}{s^3 + 4s}$ (b) $\dfrac{3}{s^2 - 4s - 5}$ (c) $\dfrac{8e^{-3s}}{s(s^2 + 4)}$

The denominator of our function in part (a) is a cubic, whose graph cannot be obtained via a shift of any *quadratic* function. From Calculus, we learn there is a partial fractions expansion of the form

$$\frac{8}{s(s^2+4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{A(s^2+4) + (Bs + C)s}{s(s^2+4)} = \frac{(A+B)s^2 + Cs + 4A}{s(s^2+4)}$$

Equating coefficients for the various powers of s (and using linear algebra?), we discover that $A = 2$, $B = -2$ and $C = 0$, so

$$\mathcal{L}^{-1}\left[\frac{8}{s^3+4s}\right] = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{2s}{s^2+4}\right] = 2\mathcal{L}^{-1}\left[\frac{1}{s}\right] - 2\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = 2 - 2\cos(2t).$$

The demoninator of the function in part (b) is quadratic, but reducible —i.e., it has real roots, exhibited by the fact that it factors

$$s^2 - 4s - 5 = (s-5)(s+1),$$

revealing roots (-1) and 5. (The quadratic formula would also reveal these *real* roots.) By using partial fraction expansion, we can turn function into the sum of functions with denominators which are 1^{st} degree polynomials:

$$\frac{3}{s^2-4s-5} = \frac{A}{s-5} + \frac{B}{s+1} = \frac{A(s+1) + B(s-5)}{(s-1)(s+5)} = \frac{(A+B)s + (A-5B)}{(s-1)(s+5)}.$$

Equating coefficients of s^1 and s^0, we can solve to find $A = 1/2, B = -1/2$. Thus,

$$\mathcal{L}^{-1}\left[\frac{3}{s^2-4s-5}\right] = \mathcal{L}^{-1}\left[\frac{1/2}{s-5} - \frac{1/2}{s+1}\right] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-5}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s-(-1)}\right] = \frac{1}{2}e^{5t} - \frac{1}{2}e^{-t}.$$

The function in part (c) is almost identical to the one in part (a), but for the exponential factor e^{-3s}. (Think Theorem B!) Piggy-backing on our answer to part (a), we obtain

$$\mathcal{L}^{-1}\left[\frac{8e^{-3s}}{s^3+4s}\right] = H(t-3)\,[2 - 2\cos(2(t-3))] = 2H(t-3) - 2H(t-3)\cos(2(t-3)).$$

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A caution concerning the use of Theorem B to find a Laplace transform

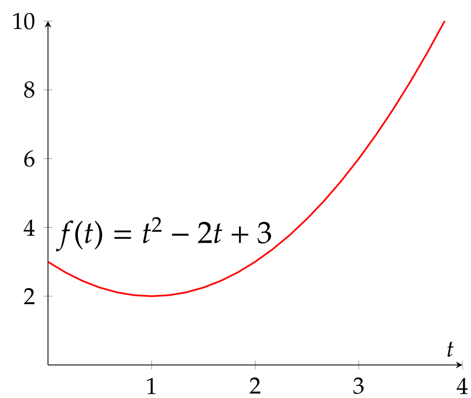
We have noted that Theorems A and B are indispensible when finding inverse Laplace transforms (going from $F(s)$ to $f(t)$), not for the reverse. That is not the same as saying the theorems are not *useful* for finding $F(s)$ from $f(t)$. Look back at Example 1, and check that the theorems provide faster ways of obtaining the answers.

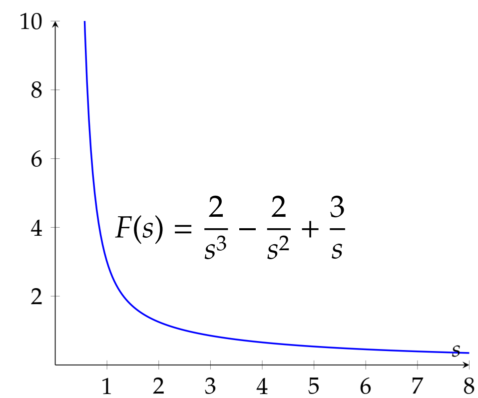
However, it is important to understand that, for a given $f(t)$, Theorem B does *not* address taking the Laplace transform of a "switched on" version of $f(t)$, but rather a "switched on and shifted" version.

Example 4:

Suppose $f(t) = t^2 - 2t + 3$. Then

$$\mathcal{L}[f(t)] \;=\; \mathcal{L}\left[t^2\right] - 2\mathcal{L}[t] + 3\mathcal{L}\left[t^0\right] \;=\; \frac{2}{s^3} - \frac{2}{s^2} + \frac{3}{s} \;=\; F(s).$$

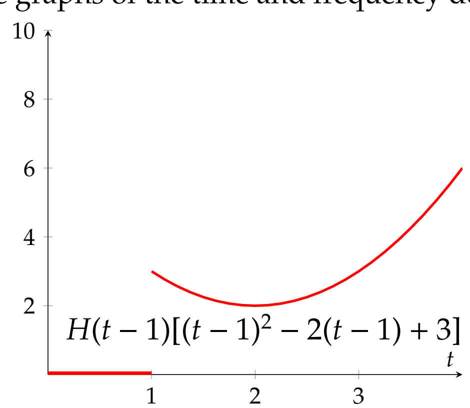


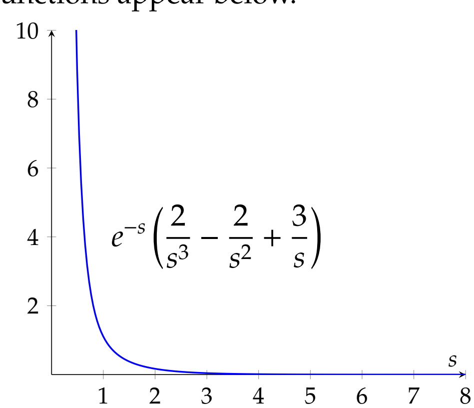


Theorem B makes it relatively easy to find the Laplace transform of $H(t-1)\, f(t-1) = H(t-1)\,[(t-1)^2 - 2(t-1) + 3]$, which has a graph like f but shifted right 1 unit and shifted on at time $t = 1$. By Theorem B,

$$\mathcal{L}[H(t-1)\, f(t-1)] \;=\; e^{-s}\left(\frac{2}{s^3} - \frac{2}{s^2} + \frac{3}{s}\right).$$

The graphs of the time and frequency domain functions appear below.





Since $(t-1)^2 - 2(t-1) + 3 = t^2 - 2t + 1 - 2t + 2 + 3 = t^2 - 4t + 4$, the graph on the left could have been labeled $H(t-1)\,(t^2 - 4t + 4)$, and the graph on the right is $\mathcal{L}\left[H(t-1)\,(t^2 - 4t + 4)\right]$.

Now, suppose what we desired was actually the Laplace transform of $H(t-1)\, f(t) = H(t-1)\,(t^2 - 2t + 3)$, whose graph is depicted at left below. We can only use Theorem B to find it if we find the formula for the function g for which $g(t-1) = f(t)$; that is,

$$g(t) \;=\; f(t+1) \;=\; (t+1)^2 - 2(t+1) + 3 \;=\; t^2 + 2t + 1 - 2t - 2 + 3 \;=\; t^2 + 2,$$

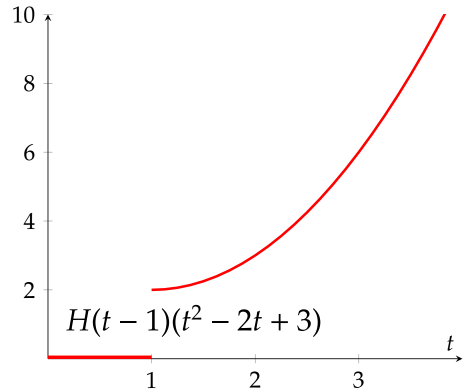
the function obtained shifting f one unit to the *left*. Since

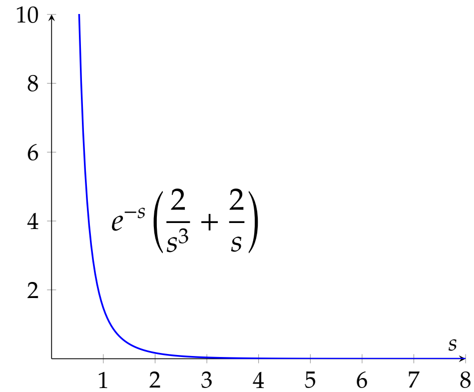
$$\mathcal{L}[g(t)] \;=\; \mathcal{L}\left[t^2 + 2\right] \;=\; \mathcal{L}\left[t^2\right] + 2\mathcal{L}[1] \;=\; \frac{2}{s^3} + \frac{2}{s},$$

then

$$\mathcal{L}[H(t-1)\,f(t)] \;=\; \mathcal{L}[H(t-1)\,g(t-1)] \;=\; e^{-s}\left(\frac{2}{s^3} + \frac{2}{s}\right),$$

whose graph appears at right below.





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Exercises

1. Graph the function and find its Laplace transform.
   (a) $f(t) = t - H(t-1)\,(t-1)$ (b) $f(t) = H\left(t - \frac{\pi}{4}\right)\cos\left(t - \frac{\pi}{4}\right)$

   (c) $f(t) = \begin{cases} 0, & t < 3 \\ t^2 + 3t - 8, & t \geq 3 \end{cases}$ (d) $f(t) = \begin{cases} 0, & t < \pi \\ t - \pi & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}$

   (e) $f(t) = e^{3t}\sin(4t)$ (f) $f(t) = 4e^{-2(t-5)}H(t-5)\,(t-5)^2$

   [Note: In the particular case of part (d), you may want to try it both writing it as a series of "shifted, switched-on" functions *and* directly from the definition of Laplace transform, and decide which you think is easier.]

2. Find the inverse Laplace transform for each function.
   (a) $F(s) = \dfrac{2(s-1)}{s^2 - 2s + 2}$ (b) $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$

   (c) $F(s) = \dfrac{4}{s^2 - 4}$ (d) $F(s) = \dfrac{4}{(s-2)^4} + \dfrac{e^{-2s}}{s^2 + s - 2}$

   (e) $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$ (f) $F(s) = \dfrac{s-2}{s^2 - 4s + 3}$