MATH 231, Worksheet
Finding inverse Laplace transforms Solutions

1. Using partial fraction expansion, we have

$$\frac{1}{s^2(s^2+4)} \;=\; \frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+4}.$$

Multiplying through by the lowest commond denominator $s^2(s^2+4)$, we get

$$1 \;=\; As(s^2+4)+B(s^2+4)+s^2(Cs+D), \tag{1}$$

an equation which must hold for all s. In particular, at $s=0$ we get

$$1 \;=\; 4B \qquad \Rightarrow \qquad B=\frac{1}{4}.$$

Thus, equation (1) becomes

$$1 \;=\; As(s^2+4)+\frac{1}{4}(s^2+4)+s^2(Cs+D),$$

or

$$0 \;=\; (A+C)s^3+\left(\frac{1}{4}+D\right)s^2+4As.$$

Equating coefficients for the various powers of s on both sides, we have

$$A+C \;=\; 0, \quad \frac{1}{4}+D \;=\; 0, \quad \text{and} \quad 4A \;=\; 0,$$

which means

$$\begin{aligned} F(s) &= \frac{1}{4}\,\frac{1}{s^2}-\frac{1}{4}\,\frac{1}{s^2+4} \\ &= \frac{1}{4}\,\frac{1}{s^2}-\frac{1}{8}\,\frac{2}{s^2+4} \\ &= \frac{1}{4}\,\mathcal{L}\{t\}-\frac{1}{8}\,\mathcal{L}\{\sin 2t\} \\ &= \mathcal{L}\left\{\frac{1}{4}\,t-\frac{1}{8}\,\sin 2t\right\}. \end{aligned}$$

Thus, $f(t)=\frac{1}{4}\,t-\frac{1}{8}\,\sin 2t$.

2. The denominator of this F does not factor (unless we use its complex roots). Instead, we complete the square on the bottom.

$$\begin{aligned} F(s) &= \frac{s}{s^2+6s+11} \\ &= \frac{s}{s^2+6s+9+2} \\ &= \frac{s}{(s+3)^2+2} \end{aligned}$$

$$\begin{aligned}
&= \frac{s+3-3}{(s+3)^2+2} \\
&= \frac{s+3}{(s+3)^2+2} - \frac{3}{(s+3)^2+2} \\
&= \left.\frac{s}{s^2+2}\right|_{s\mapsto s+3} - \frac{3}{\sqrt{2}} \left.\frac{\sqrt{2}}{s^2+2}\right|_{s\mapsto s+3}.
\end{aligned}$$

Thus,

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{ \left.\frac{s}{s^2+2}\right|_{s\mapsto s+3} \right\} - \frac{3}{\sqrt{2}} \mathcal{L}^{-1}\left\{ \left.\frac{\sqrt{2}}{s^2+2}\right|_{s\mapsto s+3} \right\} \\
&= e^{-3t}\cos\sqrt{3}t - \frac{3}{\sqrt{2}} e^{-3t}\sin\sqrt{3}t.
\end{aligned}$$

3. The only way this F differs from the one in Problem 2 is in the presence of e^{-s}. Using what we already know about the inverse Laplace transform of $s/(s^2+6s+11)$ from that problem, we have

$$f(t) = u_1(t)\left[e^{-3(t-1)}\cos\left(\sqrt{3}(t-1)\right) - \frac{3}{\sqrt{2}} e^{-3(t-1)}\sin\left(\sqrt{3}(t-1)\right)\right].$$

4. While the denominator in the second term does factor (making partial fractions possible), here I use completing the square again. To be specific,

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{ \frac{1}{(s-1)^3} + \frac{1}{s^2+2s-8} \right\} \\
&= \frac{1}{2}\mathcal{L}^{-1}\left\{ \frac{2}{(s-1)^3} \right\} + \mathcal{L}^{-1}\left\{ \frac{1}{s^2+2s+1-9} \right\} \\
&= \frac{1}{2}\mathcal{L}^{-1}\left\{ \left.\frac{2}{s^3}\right|_{s\mapsto s-1} \right\} + \mathcal{L}^{-1}\left\{ \frac{1}{(s+1)^2-9} \right\} \\
&= \frac{1}{2}t^2e^t + \mathcal{L}^{-1}\left\{ \left.\frac{1}{s^2-9}\right|_{s\mapsto s+1} \right\} \\
&= \frac{1}{2}t^2e^t + \frac{1}{3}\mathcal{L}^{-1}\left\{ \left.\frac{3}{s^2-9}\right|_{s\mapsto s+1} \right\} \\
&= \frac{1}{2}t^2e^t + \frac{1}{3}e^{-t}\sinh 3t.
\end{aligned}$$

5. The relationship between this F and the one from Problem 4 is similar to that between the ones given in Problems 3 and 2. Using the answer from Problem 4, we have

$$f(t) = \frac{1}{2}u_2(t)(t-2)^2e^{t-2} + \frac{1}{3}u_1(t)e^{1-t}\sinh(3t-3).$$

6. Here

$$f(t) = \mathcal{L}^{-1}\left\{ \frac{e^{-\pi s/2}}{s^2+9} \right\}$$

$$\begin{aligned}
&= \frac{1}{3}\,\mathcal{L}^{-1}\left\{e^{-\pi s/2}\,\frac{3}{s^2+9}\right\} \\
&= \frac{1}{3}\,\mathcal{L}^{-1}\left\{e^{-\pi s/2}\,\mathcal{L}\left\{\sin 3t\right\}\right\} \\
&= \frac{1}{3}\,u_{\pi/2}(t)\sin\left(3t-\frac{3\pi}{2}\right).
\end{aligned}$$

Incidentally, using the trigonometric identity

$$\sin(A-B) \;=\; \sin A\cos B - \cos A\sin B,$$

we get

$$f(t) \;=\; \frac{1}{3}\,u_{\pi/2}(t)\left[\sin 3t\cos\frac{3\pi}{2} - \cos 3t\sin\frac{3\pi}{2}\right] \;=\; \frac{1}{3}\,u_{\pi/2}(t)\cos 3t.$$