Top Algebra Errors Made by Calculus Students

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Preface

In recent years, it has been observed that the average student enrolled in an introductory calculus course at the college level is not as adept as she once was in her prerequisite algebra/precalculus skills. Whether this comparison with the past is accurate or not, it is certainly true that the average student makes many errors of an algebraic (rather than a calculus) nature, and this serves only to divert her focus from the subject at hand.

To address the issue of poor algebraic skills, many writers of calculus texts now include a preliminary chapter for the review of precalculus concepts. While, in theory, such a review seems a good idea, in practice, the benefits, at least in the opinion of some, are dubious. In the ideal setting such a review would get ample time (with some students getting advised to take an entire precalculus course at the college level before entering calculus); instead, it is usually clear that the teacher is trying to spend the minimal time necessary so as to arrive at the calculus as early in the semester as possible.

This document was written, both out of experiences that have led to my abandoning an algebra review at the start of first semester calculus sections I teach, and out of certain realities at my institution. As to the former, it is my experience that a two-week review of precalculus is of little value. One might think that the stronger students might be bored for a couple of weeks while the struggling ones gratefully soak in the understanding of precalculus that has previously eluded them. In my experience, it works the other way around. The strong students — the ones who would do relatively well with no review — are the most engaged, while the weaker ones are buoyed up by a false confidence bred out of a sense of familiarity with the topics. For such students there are two rude awakenings to come at the end of the review: the score on the exam testing their knowledge of precalculus, and the very unfamiliar and difficult (to some more than others) concepts of calculus, now thrown more quickly at them because several weeks were given over to review. It is a discouraging way to begin a course, and many students never get their heads above water again during the semester. In addition to experiences such as these, I have a certain sympathy for the point of view that students who have enrolled in a calculus course should get just that, not some hybrid course that is inappropriate both for those who have strong algebra skills, and for those whose lack of algebraic facility calls for a more extensive study of precalculus.

As to the realities at my institution, recent changes have led to the addition, rather than elimination, of certain topics which had not formerly been in the first year of calculus. Roughly speaking, two and a half semesters have been compressed into two. Many of our students in first-year calculus are studying to be engineers, and these changes are the result of the attempts on the part of the Mathematics Department to accomodate the wishes of our Engineering Department, while maintaining a calculus sequence that is coherent and mathematically sound. As one might expect, we have tried to make the total course content for both semesters of calculus comparable to what it was before the change; we have increased breadth at the expense of depth. Nevertheless, several weeks of review at the start of a course seems a more remote possibility than before.

It is not my response to ignore the sometimes-incomplete backgrounds of our students, but to deal with the issue differently than with an initial precalculus review. I usually take a just-in-time approach, providing a short treatise on various precalculus concepts at the moments they become relevant in our discussion of calculus. For instance, I usually treat the subject of inverses about the time one discusses the chain rule, inverse trigonometric functions, or logarithms and exponentials. Still, the day-to-day algebraic errors which students make, such as the assumption that all functions behave linearly, are not directly addressed by a review of the most important precalculus concepts, whether this review comes all-at-once at the beginning of the course, or in short bursts interspersed among calculus topics. It is for these remaining pervasive errors that I have written this piece.

To be straight about it, its purpose is two-fold. One purpose is as mentioned above, to provide a detailed discussion of some of the most common algebraic errors. The other is to reduce the amount of writing a conscientious grader feels compelled to do. I achieve the latter by giving names and 3 or 4-letter acronyms to the error types. When one of these errors is seen, a grader need only write the acronym, not a long discussion of the error. The student who receives the abbreviated 4-letter comment is not shortchanged in the least. She may turn to this document, look up the acronym, read the accompanying discussion, and benefit from it just as much (more?) as she would have from a detailed comment.

Of course, for this grading/commenting system to be successful, students must have access to this document. It currently is available on the web at

http://www.calvin.edu/∼scofield/courses/materials/tae/

Perhaps the bigger hurdle is that teachers must take the time to read and note which errors are addressed herein, along with the acronym used for each. It may take a nontrivial amount of time on the part of the instructor to get accustomed to all of these. Still, I believe the time invested in the beginning should, in the long run, reduce an instructor's grading time.

Please address comments on this document — both those intended to improve discussion of errors already addressed and those indicating an error which you think should be addressed — to the author. Also, if you use this document, a vote of confidence by way of a quick email would be appreciated.

Linear Function Behavior (LFB)

Lines through the origin are peculiar in that they have an expression of the form $f(x) = mx$, where m is a constant (the *slope*). This formula makes possible the following "additive" property:

$$f(x_1 + x_2) = m(x_1 + x_2) = mx_1 + mx_2 = f(x_1) + f(x_2).$$

For the particular choices of $x_1 = x^2$ and $x_2 = 2x$, we would have

$$f(x^2 + 2x) = f(x^2) + f(2x) = f(x^2) + 2f(x).$$

Despite students' tendancies to treat every function as additive, **other functions just do not have this property**. Typical mistakes made include:

$$\sqrt{3x^2 + 2x} = \sqrt{3x^2} + \sqrt{2x} \qquad \text{LFB}$$

$$(x+2)^3 = x^3 + 8 \qquad \text{LFB}$$

$$\ln(2x-1) = \ln(2x) - \ln 1 \qquad \text{LFB}$$

$$\frac{1}{x+5} = \frac{1}{x} + \frac{1}{5} \qquad \text{LFB}$$

Cancelling Everything in Sight (CES)

Seeing a complicated fraction become less ugly as elements are cancelled from both the numerator and denominator can be something of an enjoyable experience. One's first exposure to this magical process usually comes in grade school when reducing fractions, such as

$$\frac{52}{30} = \frac{\not{2} \cdot 26}{\not{2} \cdot 15} = \frac{26}{15}.$$

High school algebra classes build upon this, showing us that we may also cancel expressions involving variables, as in

$$\frac{(x+2)(x+4)}{(2x-1)(x+2)} = \frac{\cancel{(x+2)}(x+4)}{(2x-1)\cancel{(x+2)}} = \frac{x+4}{2x-1}.$$

What some students do not notice is that these cancellations only are performed once the numerator and denominator are *factored*. Factoring a numerator (or denominator) turns it into an expression which is, at its top level, held together by multiplication. For instance, in the expressions

$\dfrac{x(3-x)}{5(x+2)}$	numerator and denominator are factored,
$\dfrac{(3x+2)(x-6)}{5x^2+2x}$	numerator is factored, denominator is not, except as $(1)(5x^2+2x)$
$\dfrac{x(x+1)-6}{2x^5}$	denominator is factored, numerator is not,
$\dfrac{x^2-3x+5}{2x^3-1}$	neither numerator nor denominator is factored.

To be sure that one performs valid cancellations only, it is necessary to

- be patient, making sure to factor numerator and denominator first, and cancelling only those factors common to both, and
- accept that many times no factorization is possible, at least none that leads to a common, cancellable factor.

With this in mind, cancellations such as those below may only be labelled instances of someone "cancelling everything in sight", with no attention given to the discussion above, and having no validity whatsoever.

$$\frac{3x^2+2x-1}{2x-x^2} = \frac{3\cancel{x^2}+2x-1}{2x-\cancel{x^2}} = \frac{3+2x-1}{2x-1} = \frac{2+\cancel{2x}}{\cancel{2x}-1} = \frac{2}{-1} = -2.$$ CES

Any attempt to simplify the original fraction (rational expression) should start with factoring:

$$\frac{3x^2+2x-1}{2x-x^2} = \frac{(3x-1)(x+1)}{x(2-x)},$$

at which stage we see that there is no matching factor between those of the numerator — namely, $(3x-1)$ and $(x+1)$ — and those of the denominator — (x) and $(2-x)$. Factoring, in this case, did not lead to any cancelling, as is often the case.

Confusing Negative and Fractional Exponents (CNFE)

Students can make a variety of mistakes when it comes to working with exponents. Two of the most common are **Multiplying Exponents that should be Added (MEA)**, and **Adding Exponents that should be Multiplied (AEM)**. This section does not deal with either of these, but rather with a problem that some students have applying two basic rules about exponents, the ones concerning reciprocals and roots. Specifically, these are

$$\frac{1}{x^m} = x^{-m} \qquad \text{and} \qquad \sqrt[q]{x^p} = x^{p/q},$$

respectively, where the understanding is that a square root ($\sqrt{\ }$) is to be taken as ($\sqrt[2]{\ }$).

The first of these says that a *factor* of the denominator (see the discussion on **CES**) raised to a power (be it positive or negative) may be written as a factor to the oppositite power of the numerator (i.e., a (-2) power becomes $(+2)$, a $(3/4)$ power becomes $(-3/4)$). The only change is to the *sign* of the exponent. An example of a *valid* application of this rule is

$$\frac{3}{2x^3} = \frac{3}{2}x^{-3} \qquad \text{or} \qquad 3(2)^{-1}x^{-3}.$$

The second rule shows how to write a root as a power, which can be especially helpful in calculus when a derivative is desired. Things like

$$\sqrt[3]{3x^2} \quad \text{may be written as} \quad 3^{1/3}x^{2/3},$$

$$\sqrt[4]{4(x-7)} \quad \text{may be written as} \quad [2^2(x-7)]^{1/4} = 2^{1/2}(x-7)^{1/4},$$

$$\sqrt{\frac{5x}{x-2}} \quad \text{may be written as} \quad \frac{\sqrt{5}x^{1/2}}{(x-2)^{1/2}}.$$

Some students seem to confuse these two rules. The main errors seem to come from students trying to reciprocate the wrong thing

$$\frac{1}{x^2} = x^{1/2}, \qquad \text{CNFE}$$

$$\frac{2}{x^{1/2}} = 2x^2, \qquad \text{CNFE}$$

or from students putting a minus in when none is required

$$\sqrt[3]{x^2} = x^{-2/3}, \qquad \text{CNFE}$$

$$\frac{3}{\sqrt{2x-1}} = \frac{3}{(2x-1)^{-1/2}} = 3(2x-1)^{1/2}. \qquad \text{CNFE}$$

Multiplication Ignoring Powers (MIP)

Another law of exponents frequently misunderstood by students is

$$(a^m)(b^m) = (ab)^m.$$

This means that such statements as

$$\begin{aligned} 4x^2 &= 2^2x^2 = (2x)^2, \qquad \text{and} \\ 3\sqrt{x} &= 9^{1/2}x^{1/2} = (9x)^{1/2} = \sqrt{9x}. \end{aligned}$$

are correct. But many students ignore the significance of having identical powers in these multiplications. They make statements like the following, all of which are *incorrect*:

$$2x^{1/2} = \sqrt{2x}, \qquad \text{MIP}$$

$$-(3x)^2 = 9x^2, \qquad \text{MIP}$$

$$3(x+1)^2 = (3x+3)^2, \qquad \text{MIP}$$

$$\frac{3}{2x^2} = 3(2x)^{-2}. \qquad \text{MIP}$$

Equation Properties for Expressions (EPE)

Early on in one's high school algebra courses one learns several properties of equality — namely

- **Addition/Subtraction Property of Equality**: One may add/subtract the same quantity to/from both sides of a given equation, and the solutions of the resulting equation will be the same as those of the original (given) one.

- **Multiplication/Division Property of Equality**: One may multiply/divide both sides of a given equation by the same **nonzero** quantity, and the solutions of the resulting equation will be the same as those of the original (given) one.

Notice that both of these properties pre-suppose that we start with an *equation*, usually one we are supposed to solve (say, for x). These properties are helpful in achieving that goal, as in.

Solve $3x - 1 = 7$:

Add 1 to both sides:	$3x = 8$
Divide both sides by 3:	$x = \dfrac{8}{3}$

or,

Solve $\dfrac{4x+5}{3x^2+1} = 0$:

Multiply both sides by the never-zero quantity $(3x^2+1)$:	$4x + 5 = 0$
Subtract 5 from both sides:	$4x = -5$
Divide both sides by 4:	$x = -5/4.$

In contrast, these are not, generally speaking, properties one uses when trying to *simplify* an expression. (There are exceptions to this, such as in the simplifying of $\int e^x \sin x\, dx$ and $\int e^x \cos x\, dx$ using integration by parts, but these are relatively rare.) Students asked to find the derivative of

$$f(x) := \frac{3x}{2x-1}$$

may find it easier to work with the function

$$\frac{3x}{2x-1} \cdot (2x-1), \qquad \text{or} \qquad 3x,$$

but they shouldn't be under any illusions that $3x$ and $3x/(2x-1)$ are the same functions, nor that they have derivatives that are equal. If one is *simplifying* an expression like

$$\frac{12/p - 5}{3p},$$

it may be tempting to multiply by p, which gives

$$\frac{12/p-5}{3p} \cdot p \;=\; \frac{12/p-5}{3p} \cdot \frac{p}{1} \;=\; \frac{(12/p-5)p}{3p} \;=\; \frac{12-5p}{3p},$$

but, of course, multiplying by p changed the expression. One must both multiply *and divide* by p (equivalent to saying that we're multiplying by 1) if the expression is going to remain the same (but hopefully simplified):

$$\frac{12/p - 5}{3p} \cdot \frac{p}{p} = \frac{12 - 5p}{3p^2}.$$

Another example is in simplifying the difference quotient $\dfrac{\sqrt{2+x+h} - \sqrt{2+x}}{h}$.

$$\begin{aligned}
\frac{\sqrt{2+x+h} - \sqrt{2+x}}{h} &= \frac{\sqrt{2+x+h} - \sqrt{2+x}}{h} \cdot \frac{\sqrt{2+x+h} + \sqrt{2+x}}{\sqrt{2+x+h} + \sqrt{2+x}} \\
&\qquad \text{(really multiplication by 1)} \\
&= \frac{(\sqrt{2+x+h} - \sqrt{2+x})(\sqrt{2+x+h} + \sqrt{2+x})}{h(\sqrt{2+x+h} + \sqrt{2+x})} \\
&= \frac{2+x+h-(2+x)}{h(\sqrt{2+x+h} + \sqrt{2+x})} \\
&= \frac{h}{h(\sqrt{2+x+h} + \sqrt{2+x})} \\
&= \frac{1}{\sqrt{2+x+h} + \sqrt{2+x}},
\end{aligned}$$

which cannot be further simplified.

Multiplication Without Parentheses (MWP)

The discussion here necessarily must begin with an appeal to the **order of algebraic operations** (**OO**). These are rules of hierarchy as to which operations to perform 1st, 2nd, etc. when an algebraic expression requires more than one operation be performed. There are three levels of hierarchy:

(1) powers,

(2) multiplication and division, and

(3) addition and subtraction.

When faced with an expression like the one below that has both an addition and a multiplication in it, the order of operations dictates that the multiplication be performed first:

$$2 + 3 \cdot 5 \quad \text{is} \quad 17, \quad \text{not} \quad 30.$$

The levels above do not give the whole story, however. For instance, what if an expression has both an addition and a subtraction, operations which appear at the same level? The answer here is that operations appearing on the same level are always performed left-to-right:

$$2+3-5 \quad \text{is} \quad 0, \qquad 2-3+5 \quad \text{is} \quad 4, \qquad \text{and} \qquad 2/3\cdot 5 \quad \text{is} \quad \frac{10}{3}.$$

Also, one may use parentheses to override these rules. Things in parentheses are performed before things outside of those parentheses, starting from the inside and working out. So

$$2-(3-(2-6)) \quad \text{is} \quad -5,$$

while

$$2-(3-2-6) \quad \text{is} \quad 7$$

and

$$2-3-2-6 \quad \text{is} \quad -13.$$

These order of operations apply to expressions involving variables as well. Thus

$$x\cdot 2x-7 \qquad \text{is not the same as} \qquad x(2x-7),$$

$$3-x/x^2 \qquad\qquad (3-x)/x^2.$$

In this light, acceptable notation for the product of two expressions like (3) and $(-5x^2)$ is

$$(3)(-5x^2) \qquad \text{or, more simply} \qquad -15x^2,$$

not, as so many students write,

$$3\cdot -5x^2. \qquad \textbf{MWP}$$

Frivolous Parentheses (FP)

There really isn't an error, *per se*, with using too many parentheses. Nevertheless, students who consistently employ more parentheses than needed are demonstrating as much of a lack of understanding of the order of algebraic operations as those who use too few. Expressions such as

$$\left(\frac{(2x^2+3x)}{(x-1)}\right) \qquad \text{look simpler as} \qquad \frac{2x^2+3x}{x-1},$$

$$\left(\frac{(2x^2+3x)}{(x-1)}\right)^3 \qquad\qquad \left(\frac{2x^2+3x}{x-1}\right)^3 \quad \text{or} \quad \frac{(2x^2+3x)^3}{(x-1)^3},$$

$$(3)\left((x^3-6)(-\sqrt{x})\right) \qquad\qquad -3(x^3-6)\sqrt{x}$$

$$\frac{(3-x)}{(x^3)} \qquad\qquad \frac{3-x}{x^3} \quad \text{or} \quad (3-x)/x^3.$$

Undo Multiplication with Division (UMD); also, **Undo Addition with Subtraction (UAS)**

The properties of equality that were mentioned earlier are, by some students, implemented incorrectly even when the situation calls for their use. For instance, when solving an equation like

$$3x + 7 = 4,$$

two steps are called for:

$3x = -3$ (**subtraction prop. of equality**; 7 subtracted from both sides)

and

$x = -1$ (**division prop. of equality**; both sides divided by 3).

Notice that, in the expression $(3x + 7)$, the order of operations dictates that the multiplication by 3 comes before the addition of 7, and the "undoings" of these processes — the subtraction of 7 and the division by 3 — were carried out in reverse order. That is not to say that we had to undo things in this order, but students who use a different sequence often make the following error. Dividing by 3, they often neglect the fact that *all* terms on both sides are to be divided by 3. In other words, after dividing by 3 they write

$$x + 7 = \frac{4}{3} \qquad \text{instead of} \qquad x + \frac{7}{3} = \frac{4}{3}.$$

They are too set on the idea that they will be subtracting 7 from both sides to realize that, having divided by 3 first, it is not 7, but rather 7/3, which must be subtracted, giving the same answer $x = -1$ as before. One other note is in order here. If parentheses appear in an equation such as

$$3(x - 1) = 5,$$

then the order of operations are preempted (the subtraction within the parentheses comes before the multiplication by 3). In solving for x, we may of course, distribute the 3, thereby eliminating the parentheses and making the problem appear similar to the last one discussed. Even fewer steps are required if one just "undoes" the multiplication and subtraction in their opposite order:

$x - 1 = \frac{5}{3}$ (**division prop. of equality**; dividing both sides by 3),

and then

$x = \frac{8}{3}$ (**addition prop. of equality**; adding 1 to both sides).

Now let us return to the equation

$$3x + 7 = 4,$$

and investigate the more telling errors that gave the titles **UMD** and **UAS** to this section. Some students recognize the need for two steps (like those carried out when this equation was being considered above) to isolate x, but have little feel for which operations will achieve this. For instance, realizing that, like the 4 on the right-hand side of the equation, 7 is a "non-x" term, a student may write

$$3x = \frac{4}{7}, \qquad \text{UAS}$$

misunderstanding that she has subtracted 7 on the left side, but divided by 7 on the other side. *The original equation and the new one no longer have the same solution* as a result. The same student may then recognize that she needs to move the 3 over to the other side. Since the 3 is multiplied by the x, she should "undo" this by dividing both sides by 3. But she may (wrongly) write

$$x = \frac{4}{7} - 3 = -\frac{17}{7}, \qquad \text{UMD}$$

having divided on the left but subtracted on the right. Again, the solution ($x = -17/7$) is different from the one that solved the original equation $3x + 7 = 4$, namely $x = -1$.

Worse still is when a student thinks he can solve in one step (that is, take care both of the multiplication by 3 and the addition of 7 via one operation). Such a student may write something like

$$3x + 7 = 4 \quad \Rightarrow \quad x = \frac{4}{3+7}. \qquad \text{UAS /UMD}$$

Again, the answer this student gets, $x = \frac{2}{5}$, is different than the correct one $x = -1$.

Misunderstood Relationship between Roots and Zeros (MRRZ)

Much of one's mathematical experience prior to the calculus is spent in solving equations. There are the kind of equations, known as *identities*, where every number in the domain is a solution. The equation

$$(x - 2)(2x + 3) = 2x^2 - x - 6,$$

is such an identity. It is not this, but the other type of equation, known as a *conditional equality*, that one learns to solve, precisely because solutions, also known as *roots*, of conditional equalities are not everywhere to be found. Often there are very few numbers, perhaps even none at all, which make a conditional equality true.

Another fact about conditional equalities is that comparatively few of them may be solved exactly. Leaps in technology have made it commonplace for students, with the purchase of a handheld calculator, to have at their fingertips powerful graphing capability and numerical methods for finding approximate solutions to many, perhaps even most conditional equalities. This does not mean that one should forego learning the algebraic techniques which lead to exact solutions, thinking that deftness in pushing the right pair of buttons is an appropriate substitute for the thought processes such algebraic methods introduce. Still, there is added value in the knowledge one gets by investigating graphical methods. By these methods one comes to think of the solutions of, say,

$$3x^2 - 2x = 2x^3 + x^2 - 1$$

as the x-values of points of intersection between the graphs of the two functions

$$y = 3x^2 - 2x \qquad \text{and} \qquad y = 2x^3 + x^2 - 1.$$

As another example, solutions of the equation

$$x^2 + 5x = -6$$

would be found at points of intersection between the graphs of

$$y = x^2 + 5x \quad \text{(a parabola)} \qquad \text{and} \qquad y = -6 \quad \text{(a horizontal line)}.$$

It is in solving equations like this latter one that students become confused. What some students do is the following:

$$\begin{aligned} x^2 + 5x = -6 \qquad &\Rightarrow \qquad x(x+5) = -6 \quad \text{(factor the left side)} \\ &\Rightarrow \qquad x = 0 \;\text{ or }\; x + 5 = 0 \quad \text{(set factors equal to zero)} \\ &\Rightarrow \qquad x = 0 \;\text{ or }\; x = -5. \end{aligned}$$

MRRZ

In mathematical terms, the student who does these steps has found the *zeros* of $f(x) = x^2 + 5x$; that is, the values for x which make the output of f be zero. There are several ways to see that this work is wrong. One way to see it is that, in the equation, we want values of x whose output value is (-6), not zero. Another angle which reveals the errors is the one that notes that, while there are a lot of pairs of numbers which may be multiplied to give (-6) — (-1) and 6, 12 and (-1/2), 55 and (-6/55) are three such pairs — one thing which we can say for certain is that neither of the numbers in the pair is zero, which is quite counter to the idea of setting the factors equal to zero. (Of course, neither is it enough to set the factors equal to (-6), as in

$$
\begin{aligned}
x(x+5) = -6 \quad &\Rightarrow \quad x = -6 \ \text{ or } \ x+5 = -6 \\
&\Rightarrow \quad x = -6 \ \text{ or } \ x = -11,
\end{aligned}
$$

MRRZ',

since it is not enough for either one of these conditions to hold by itself; that is, if $x = -6$ then we would need the other factor $(x + 5)$ to be equal to 1 in order for their product to be (-6), and clearly these things cannot occur at the same time.)

In summary, the error occurs in finding the zeros of a function and taking them to be the roots of the equation, when the two concepts do not coincide. There is a simple way to make them coincide. We simply make one side zero (using a valid algebraic step, of course).

$$
\begin{aligned}
x^2 + 5x = -6 \quad &\Rightarrow \quad x^2 + 5x + 6 = 0 \\
&\Rightarrow \quad (x+2)(x+3) = 0 \\
&\Rightarrow \quad x+2 = 0 \ \text{ or } \ x+3 = 0 \\
&\Rightarrow \quad x = -2 \ \text{ or } \ x = -3.
\end{aligned}
$$

The zeros of $g(x) = x^2 + 5x + 6$ are the numbers which make g equal zero, and that is exactly what we want in a solution of the equation $x^2 + 5x + 6 = 0$, so the two concepts coincide. Why do students mess this up? The most likely answer is that many are looking to do as little work as possible, and bringing the (6) over makes factoring a more difficult job (it is harder to factor $x^2 + 5x + 6$ than to factor $x^2 + 5x$); of course, the quadratic formula is an option for this case. What may help to avoid this confusion is remembering this graphical interpretation of what one is doing (still applied to the example above):

- Solutions of an equation like $x^2 + 5x = -6$ correspond to points of intersection between the two sides, considered as functions, of the equation (i.e., the function $x^2 + 5x$ and the function, in this case a constant one, (-6)).
- If the two functions are combined into one function on one side of the equation, there is still a second function, the zero function, that remains on the other side. Now we have in place of the old problem a new one (but entirely equivalent) of finding the solutions that

correspond to points of intersection between the new left-hand side (in this case $x^2 + 5x + 6$) and the new right-hand side (here zero).

- When our combining of terms has left one side of the equation zero (which, when considered as a function, has the x-axis as its graph), one may solve the equation by finding the zeros of the other side of the equation.

Multiplication Not Distributive (MND)

In precalculus/algebra we become familiar with the *distributive laws* that address interactions between multiplication and addition/subtraction. Specifically, these laws say

$$a(b + c) = ab + ac \qquad \text{and} \qquad (a + b)c = ac + bc.$$

We use these laws all the time, both in expanding

$$\begin{aligned} 3x^2(x - 2y) &= 3x^3 - 6x^2y \qquad \text{and} \\ (x - 3)(x + 7) &= (x - 3)x + (x - 3)(7) = x^2 - 3x + 7x - 21 = x^2 + 4x - 21, \end{aligned}$$

and in factoring

$$30x^2y - 12xy^2 + 3xy = 3xy(10x - 4y + 1).$$

We even use it (although we don't often think about it this way and usually don't include the middle step below) when combining like terms, as in

$$4xy - 15xy = (4 - 15)xy = -11xy.$$

The problem is when students misinterpret these laws, thinking they also say something about interactions between more than one multiplication; that is, they "invent" for themselves a law that looks something like:

$$a(b \cdot c) = (ab)(ac). \qquad \textbf{MND}$$

This clearly is false, as most would see if these were all numbers — few (though I cannot go so far as to say *no one*) would assert, say, that

$$7(3 \cdot 10) = (21)(70) = 1470. \qquad \textbf{MND}$$

But when the objects involved are expressions involving variables, the error is frequently made, such as in this case:

$$5(2x^2y^3) = (10x^2)(5y^3) = 50x^2y^3, \qquad \textbf{MND}$$

or

$$3[(x-1)(7x)] = (3x-3)(21x). \qquad \textbf{MND}$$

Poor Use of Mathematical Language (PUML)

A prerequisite skill to writing good mathematics is the ability to write well in one's native tongue. People who cannot write a complete English sentence should take remediation in English composition before reading on.

What may surprise some students is that good writing using mathematical symbols (even in the write-up of homework problems) consists of using complete sentences, setting up one's ideas clearly and then following through on the details, much as one expects from a good English essay. The language and symbols of mathematics are used just like regular English words and phrases to express ideas, albeit ideas which one would often struggle to use any other means of expressing.

Nobody studies mathematical writing as a subject. Your mathematics professor(s) got to be good writers of mathematics, if good they be, by the reading papers and books of other mathematicians, not by reading a treatise such as this one. If a book on good mathematical writing *does* exist (and there probably are a number of such books), they will say much more than I say here. I will only describe the most common example of poor mathematical writing I see when grading students' work: Using Equals as a Conjunction (**UEC**).

The word "equals" has a very specific meaning. It requires two objects, and it asserts that these two objects are the same. In mathematics, the two objects are usually quantities, like the mathematical expression $(3x + 5)$, or the number 7. Even within this tight definition, mathematical equations, as I mentioned earlier, come in two varieties: *identities* and *conditional equations*. A conditional equation is one such as the equation

$$3x + 5 = 7,$$

which is true only for particular values of x (in this case *one* particular value). In algebra courses one often sees conditional equalities in homework problems accompanied by the instruction "Solve the equation". There are some quantities that are the same regardless of the value of the variable. A familiar example is the identity

$$\sin^2 x + \cos^2 x = 1,$$

which is true no matter what real value x takes. These two types of equations encompass the two most common (and only?) valid ways to use an 'equals' (=) sign.

Consider the typical calculus problem of evaluating a limit like

$$\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4}.$$

What we are given here is not an equation, but an expression. If we begin writing a series of equalities to simplify/evaluate this expression, we will want them to be identities, as in

$$\begin{aligned} \lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4} &= \lim_{x\to 2} \frac{(x-2)(x+1)}{(x+2)(x-2)} \\ &= \lim_{x\to 2} \frac{x+1}{x+2} \\ &= \frac{3}{4}. \end{aligned}$$

The original expression, along with each of the ensuing expressions, as it turns out, are all equal to the number 3/4.

In contrast, suppose we begin with a (conditional) equation like

$$3x + 5 = 7,$$

which we are asked to solve. If a student who understands very well the discussion of **UAS** and **UMD** (found earlier in this piece) makes a mistake, she is most likely to do so writing something like

$$3x + 5 \;=\; 7 - 5 \;=\; \frac{2}{3}. \qquad \textbf{UEC}$$

Such a string of equalities asserts three things:

(i) that $3x + 5 = 2$,
(ii) that $3x + 5 = 2/3$,
(iii) and that $2 = 2/3$.

(i) and (ii) are conditional equations in their own right, but it should be clear that they do not have the same solutions as the original equation $3x + 5 = 7$ (nor does (i) have the same solution as (ii)). And (iii) has no solution at all, for it is never true. What I am really saying is that the string of equations

$$3x + 5 \;=\; 7 - 5 \;=\; \frac{2}{3} \tag{1}$$

is really three equations, and there is no common solution between them (and, even if there had been, such a solution would have no relevance to the original problem, that of solving $3x+5 = 7$). The student most likely never intended to assert these three equations in place of the original; she simply began writing out her ideas, and used an equals sign to join them together whenever it seemed some sort of 'conjunction' was required.

The student who writes (1) actually appears to have some facility in the techniques for solving linear equations, but lacks the ability to put her ideas onto paper in a meaningful fashion. One good way to express the solution of the previous equation is

$$\begin{aligned} 3x + 5 \;=\; 7 \qquad &\Rightarrow \qquad 3x \;=\; 2 \\ &\Rightarrow \qquad x \;=\; \frac{2}{3}. \end{aligned} \tag{2}$$

The symbol $\Rightarrow$ can be translated here as "which implies". Yes, (2) is more writing than (1), much in the same way the complete sentence "I am taking the train to Chicago this weekend" requires more writing than the three words "weekend, Chicago, train". A more favorable comparison is between (2) and the same ideas expressed in English words:

> If the sum of three times x and five is seven, then subtracting five from both sides and dividing by three yields the value of two-thirds for x,

or, perhaps more literally,

> Assuming that the sum of three times x and five is seven, this implies that three times x is two, and that x is two-thirds.

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