ODE Cheat Sheet for Math 331

Scalar linear first-order ODEs

Consider the ODE,

$$\dot{x} = a(t)x + f(t).$$

Setting

$$A(t) = \int a(t)\,\mathrm{d}t,$$

the general solution is given by the variation of parameters formula,

$$x(t) = c_1 \mathrm{e}^{A(t)} + \mathrm{e}^{A(t)} \int \mathrm{e}^{-A(t)} f(t)\,\mathrm{d}t.$$

In the special case that $a(t) \equiv a$, the solution becomes

$$x(t) = c_1 \mathrm{e}^{at} + \mathrm{e}^{at} \int \mathrm{e}^{-at} f(t)\,\mathrm{d}t.$$

Scalar linear second-order homogeneous ODEs

Consider the homogeneous ODE,

$$\ddot{x} + p\dot{x} + qx = 0.$$

The characteristic equation is

$$\lambda^2 + p\lambda + q = 0.$$

Let λ_1, λ_2 be the roots of the characteristic equation. If $\lambda_1 \neq \lambda_2$ are real, the homogeneous solution is

$$x_\mathrm{h}(t) = c_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathrm{e}^{\lambda_2 t}.$$

If $\lambda_1 = \lambda_2 = -p/2$, which requires $p^2 = 4q$, the homogeneous solution is

$$x_\mathrm{h}(t) = c_1 \mathrm{e}^{-pt/2} + c_2 t \mathrm{e}^{-pt/2}.$$

If $\lambda_1 = a + \mathrm{i}b$ with $b \neq 0$, the homogeneous solution is

$$x_\mathrm{h}(t) = c_1 \mathrm{e}^{at} \cos(bt) + c_2 \mathrm{e}^{at} \sin(bt).$$

Scalar linear second-order nonhomogeneous ODEs

Consider the nonhomogeneous ODE,

$$\ddot{x} + p\dot{x} + qx = f(t).$$

The solution is the sum of the homogeneous and particular solutions,

$$x(t) = x_{\mathrm{h}}(t) + x_{\mathrm{p}}(t).$$

Write the homogeneous solution as

$$x_{\mathrm{h}}(t) = c_1 x_1(t) + c_2 x_2(t),$$

and set

$$\Phi(t) = x_1(t)x_2'(t) - x_2(t)x_1'(t).$$

The particular solution is given by the variation of parameters formula,

$$x_{\mathrm{p}}(t) = -\left(\int \frac{x_2(t)f(t)}{\Phi(t)}\,\mathrm{d}t\right)x_1(t) + \left(\int \frac{x_1(t)f(t)}{\Phi(t)}\,\mathrm{d}t\right)x_2(t).$$

First-order homogeneous linear system of ODEs

Consider the first-order system,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{A}$ are λ_1, λ_2, and the associated eigenvectors are $\boldsymbol{v}_1, \boldsymbol{v}_2$. If the eigenvalues are real the general solution is,

$$\boldsymbol{x}(t) = c_1 \mathrm{e}^{\lambda_1 t}\boldsymbol{v}_1 + c_2 \mathrm{e}^{\lambda_2 t}\boldsymbol{v}_2.$$

If $\lambda_1 = a + \mathrm{i}b\,(b \neq 0)$ with associated eigenvector $\boldsymbol{v}_1 = \boldsymbol{p} + \mathrm{i}\boldsymbol{q}$, the general solution is,

$$\boldsymbol{x}(t) = c_1 \mathrm{e}^{at}\left(\cos(bt)\boldsymbol{p} - \sin(bt)\boldsymbol{q}\right) + c_2 \mathrm{e}^{at}\left(\sin(bt)\boldsymbol{p} + \cos(bt)\boldsymbol{q}\right).$$

We have the following classification of the fixed point, $\boldsymbol{x} = \boldsymbol{0}$:

(a) $\lambda_1 < 0 < \lambda_2$: unstable saddle point

(b) $\lambda_1 < \lambda_2 < 0$: stable node

(c) $0 < \lambda_1 < \lambda_2$: unstable node

(d) if $\lambda_1 = a + \mathrm{i}b\,(b \neq 0)$,

- $a < 0$: stable spiral
- $a > 0$: unstable spiral
- $a = 0$: linear center.