Math 355 Homework Problems #1

MATRIX ANALYSIS AND APPLIED LINEAR ALGEBRA, by C. Meyer

1. Find the coefficients for the cubic function $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ which allow it to pass through the four points $(0,1)$, $(1,1)$, $(2,7)$, and $(3,31)$.

2. Consider the matrix $\boldsymbol{A} \in \mathcal{M}_4(\mathbb{F})$,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -1 & -5 \\ 3 & 6 & 2 & 0 \\ -2 & -4 & 3 & 13 \\ 4 & 8 & 1 & -5 \end{pmatrix}.$$

(a) What is $\operatorname{rank}(\boldsymbol{A})$?

(b) Find all solutions to $\boldsymbol{Ax} = \boldsymbol{0}$.

3. Consider the boundary value problem,

$$y' = f(x), \quad y(0) = y(1) = 0.$$

The goal is to recast this BVP as a linear algebra problem. Pick $N \geq 1$, and discretize the unit interval via

$$x_j = jh, \ j = 0, 1, \ldots, N; \quad h = \frac{1}{N}.$$

Set

$$y_j = y(x_j), \quad f_j = f(x_j).$$

(a) Using the rule,

$$y'(x) \sim \frac{y(x+h) - y(x-h)}{2h},$$

find the matrix $\boldsymbol{D}$, and vectors $\boldsymbol{y}, \boldsymbol{f}$, so that the BVP is equivalent to $\boldsymbol{Dy} = \boldsymbol{f}$.

(b) Using the rule,

$$y'(x) \sim \frac{y(x+h) - y(x)}{h},$$

find the matrix $\boldsymbol{D}$, and vectors $\boldsymbol{y}, \boldsymbol{f}$, so that the BVP is equivalent to $\boldsymbol{Dy} = \boldsymbol{f}$.

(c) For which formulation is $\boldsymbol{D}$ skew-Hermitian?

4. Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$ be a square matrix. Show that:

(a) $\boldsymbol{A} + \boldsymbol{A}^{\mathrm{H}}$ is a Hermitian matrix

(b) $\boldsymbol{A} - \boldsymbol{A}^{\mathrm{H}}$ is a skew-Hermitian matrix

(c) $\boldsymbol{A}$ can be written as the sum of a Hermitian matrix and a skew-Hermitian matrix.

5. Let $\boldsymbol{A} \in \mathcal{M}_{m\times n}(\mathbb{F})$. Show that:

(a) $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}} \in \mathcal{M}_m(\mathbb{F})$ is Hermitian

(b) $\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$ is Hermitian.

6. When doing the method of undetermined coefficients in ODEs we were sometimes confronted with linear systems of the form,

$$\begin{pmatrix} \boldsymbol{C} & -\boldsymbol{I}_2 \\ \boldsymbol{I}_2 & \boldsymbol{C} \end{pmatrix}\begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix},$$

where $\boldsymbol{C} \in \mathcal{M}_2(\mathbb{F})$, and $\boldsymbol{x}_j, \boldsymbol{b}_j \in \mathbb{F}^2$ for $j = 1, 2$. Assume that $\boldsymbol{I}_2 + \boldsymbol{C}^2$ is invertible.

(a) Show that this linear system can be solved if one first solves the smaller system,

$$(\boldsymbol{C}^2 + \boldsymbol{I}_2)\boldsymbol{x}_2 = \boldsymbol{C}\boldsymbol{b}_2 - \boldsymbol{b}_1.$$

(b) If $\boldsymbol{x}_2$ is a solution to the problem in part (a), what is $\boldsymbol{x}_1$?

(c) Find the inverse of the full matrix in terms of the submatrices, $\boldsymbol{I}_2$ and $\boldsymbol{C}$.

7. A matrix, $\boldsymbol{U}$, is unitary if $\boldsymbol{U}^{\mathrm{H}} = \boldsymbol{U}^{-1}$. If $\boldsymbol{Q}, \boldsymbol{U}$ are unitary, show that $\boldsymbol{Q}\boldsymbol{U}$ is unitary.