Math 355 Homework Problems #1
Matrix Analysis and Applied Linear Algebra, by C. Meyer

1. Find the coefficients for the cubic function \( y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \) which allow it to pass through the four points \((0, 1), (1, 1), (2, 7), \) and \((3, 31)\).

2. Consider the matrix \( A \in M_4(\mathbb{F}) \),
\[
A = \begin{pmatrix}
1 & 2 & -1 & -5 \\
3 & 6 & 2 & 0 \\
-2 & -4 & 3 & 13 \\
4 & 8 & 1 & -5
\end{pmatrix}.
\]

(a) What is rank\((A)\)?
(b) Find all solutions to \( Ax = 0 \).

3. Consider the boundary value problem,
\[
y' = f(x), \quad y(0) = y(1) = 0.
\]
The goal is to recast this BVP as a linear algebra problem. Pick \( N \geq 1 \), and discretize the unit interval via
\[
x_j = jh, \quad j = 0, 1, \ldots, N; \quad h = \frac{1}{N}.
\]
Set \( y_j = y(x_j), \quad f_j = f(x_j) \).

(a) Using the rule,
\[
y'(x) \sim \frac{y(x + h) - y(x - h)}{2h},
\]
find the matrix \( D \), and vectors \( y, f \), so that the BVP is equivalent to \( Dy = f \).

(b) Using the rule,
\[
y'(x) \sim \frac{y(x + h) - y(x)}{h},
\]
find the matrix \( D \), and vectors \( y, f \), so that the BVP is equivalent to \( Dy = f \).

(c) For which formulation is \( D \) skew-Hermitian?

4. Let \( A \in M_n(\mathbb{F}) \) be a square matrix. Show that:

(a) \( A + A^H \) is a Hermitian matrix

(b) \( A - A^H \) is a skew-Hermitian matrix

(c) \( A \) can be written as the sum of a Hermitian matrix and a skew-Hermitian matrix.
5. Let $A \in \mathcal{M}_{m \times n}(\mathbb{F})$. Show that:

(a) $AA^H \in \mathcal{M}_m(\mathbb{F})$ is Hermitian

(b) $A^HA \in \mathcal{M}_n(\mathbb{F})$ is Hermitian.

6. When doing the method of undetermined coefficients in ODEs we were sometimes confronted with linear systems of the form,

\[
\begin{pmatrix}
C & -I_2 \\
I_2 & C
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
b_1 \\
b_2
\end{pmatrix},
\]

where $C \in \mathcal{M}_2(\mathbb{F})$, and $x_j, b_j \in \mathbb{F}^2$ for $j = 1, 2$. Assume that $I_2 + C^2$ is invertible.

(a) Show that this linear system can be solved if one first solves the smaller system,

$(C^2 + I_2)x_2 = Cb_2 - b_1$.

(b) If $x_2$ is a solution to the problem in part (a), what is $x_1$?

(c) Find the inverse of the full matrix in terms of the submatrices, $I_2$ and $C$.

7. A matrix, $U$, is unitary if $U^H = U^{-1}$. If $Q, U$ are unitary, show that $QU$ is unitary.