Math 355 Homework Problems #5

MATRIX ANALYSIS AND APPLIED LINEAR ALGEBRA, by C. Meyer

1. Suppose in terms of the standard basis,

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{pmatrix}.$$

Find the representation of A in terms of the basis,

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

2. Let $\boldsymbol{A} \in \mathcal{M}_{m\times n}(\mathbb{F})$ with $m \geq n$. Suppose $\boldsymbol{A} = \boldsymbol{QR}$, where the columns of $\boldsymbol{Q} \in \mathcal{M}_{m\times n}(\mathbb{F})$ are orthonormal, and $\boldsymbol{R} \in \mathcal{M}_n(\mathbb{F})$ is upper-triangular with real and positive diagonal entries.

(a) Show that $\boldsymbol{R}$ is invertible.

(b) Show that $\operatorname{rank}(\boldsymbol{Q}) = n$.

(c) Show that $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{QR}) = n$.

(d) Solve the normal equations, $\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^{\mathrm{H}}\boldsymbol{b}$, using $\boldsymbol{R}, \boldsymbol{Q}$ (but, do not use $\boldsymbol{Q}^{-1}$!).

3. $\boldsymbol{A}$ is normal if $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}} = \boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}$. If $\boldsymbol{A}$ is normal, show that $\operatorname{Col}(\boldsymbol{A})^{\perp} = \operatorname{Null}(\boldsymbol{A})$. (*Hint*: Recall that $\operatorname{Null}(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}) = \operatorname{Null}(\boldsymbol{A})$)

4. Let $S \subset \mathbb{F}^n$ be a subspace with orthonormal basis, $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$, and set $\boldsymbol{U} = (\boldsymbol{u}_1\ \boldsymbol{u}_2\ \cdots\ \boldsymbol{u}_k)$. Define the matrices,

$$\boldsymbol{P}_S = \boldsymbol{U}\boldsymbol{U}^{\mathrm{H}}, \quad \boldsymbol{P}_{S^\perp} = \boldsymbol{I}_n - \boldsymbol{P}_S.$$

Show that:

(a) $\boldsymbol{P}_S \cdot \boldsymbol{P}_S = \boldsymbol{P}_S$

(b) $\boldsymbol{P}_{S^\perp} \cdot \boldsymbol{P}_{S^\perp} = \boldsymbol{P}_{S^\perp}$

(c) $\boldsymbol{P}_S \cdot \boldsymbol{P}_{S^\perp} = \boldsymbol{P}_{S^\perp} \cdot \boldsymbol{P}_S = \boldsymbol{0}_n$

(d) $\operatorname{Null}(\boldsymbol{P}_S) = S^\perp$

(e) $\operatorname{Null}(\boldsymbol{P}_{S^\perp}) = S$

(f) $\operatorname{Col}(\boldsymbol{P}_S) = S$

(g) $\operatorname{Col}(\boldsymbol{P}_{S^\perp}) = S^\perp$.

5. Suppose,

$$S = \operatorname{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \right\} \subset \mathcal{M}_2(\mathbb{F}), \quad \boldsymbol{A} = \begin{pmatrix} -2 & 3 \\ 0 & 5 \end{pmatrix}.$$

Compute $\operatorname{proj}_S(\boldsymbol{A})$ using the inner product, $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \operatorname{trace}(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{B})$.