MATH W81: HOMEWORK #3

1. Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be an orthonormal set of vectors.

(a) Set
$$\boldsymbol{P}_1 = \boldsymbol{u}_1\boldsymbol{u}_1^{\mathrm{H}} + \boldsymbol{u}_2\boldsymbol{u}_2^{\mathrm{H}} + \cdots + \boldsymbol{u}_k\boldsymbol{u}_k^{\mathrm{H}},$$
where for any vector $\boldsymbol{x} \in \mathbb{C}^n$,
$$\boldsymbol{x}^{\mathrm{H}} = \overline{\boldsymbol{x}^{\mathrm{T}}} = (\overline{\boldsymbol{x}})^{\mathrm{T}}.$$
Show that $\boldsymbol{P}_1^2\boldsymbol{x} = \boldsymbol{P}_1\boldsymbol{x}$, where $\boldsymbol{P}_1^2 = \boldsymbol{P}_1 \cdot \boldsymbol{P}_1$.

(b) Set
$$\boldsymbol{S} = (\boldsymbol{u}_1\ \boldsymbol{u}_2\ \cdots\ \boldsymbol{u}_k).$$
If $\boldsymbol{P}_2 = \boldsymbol{S}\boldsymbol{S}^{\mathrm{H}}$, show that $\boldsymbol{P}_2^2\boldsymbol{x} = \boldsymbol{P}_2\boldsymbol{x}$.

(c) Show that $\boldsymbol{P}_1$ is a matrix of rank k (*hint*: find $R(\boldsymbol{P}_1)$).

(d) Show that $\boldsymbol{P}_2$ is a matrix of rank k (*hint*: find $R(\boldsymbol{P}_2)$).

(e) Show that $\boldsymbol{P}_1\boldsymbol{x} = \boldsymbol{P}_2\boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbb{C}^n$.

2. Suppose there exist matrices $\boldsymbol{A}, \boldsymbol{P}, \boldsymbol{D} \in \mathbb{C}^{n\times n}$ such that
$$\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{D}.$$
If $\boldsymbol{P}$ is invertible, show that $\sigma(\boldsymbol{A}) = \sigma(\boldsymbol{D})$.

3. Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ be a skew-Hermitian matrix, i.e., $\boldsymbol{A}^{\mathrm{H}} = -\boldsymbol{A}$. Show that all of the eigenvalues are purely imaginary (*hint*: consider the matrix i$\boldsymbol{A}$).

4. If
$$S = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \\ -1 \end{pmatrix}\right\},$$
find a projection matrix $\boldsymbol{P}$ such that:

(a) $\boldsymbol{P} : \mathbb{C}^4 \mapsto S$

(b) $\boldsymbol{P}^2\boldsymbol{x} = \boldsymbol{P}\boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbb{C}^4$

(c) $\boldsymbol{P}\boldsymbol{x} = \boldsymbol{0}$ for all $\boldsymbol{x} \in S^{\perp}$.