Math W81: Homework #4

1. A Hermitian matrix $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ has the spectral decomposition

$$\boldsymbol{A} = \lambda_1 \boldsymbol{G}_1 + \lambda_2 \boldsymbol{G}_2 + \cdots + \lambda_n \boldsymbol{G}_n.$$

The rank-one projection matrices $\boldsymbol{G}_j$ have as their range $\operatorname{Span}\left(\{\boldsymbol{u}_j\}\right)$, where $\boldsymbol{u}_j$ is an eigenvector associated with the eigenvalue λ_j. Show that for any integer ℓ,

$$\boldsymbol{A}^\ell = \lambda_1^\ell \boldsymbol{G}_1 + \lambda_2^\ell \boldsymbol{G}_2 + \cdots + \lambda_n^\ell \boldsymbol{G}_n.$$

2. Find the spectral decomposition of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}.$$

3. Find the QR factorization of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \\ 0 & -1 & -1 \end{pmatrix}.$$

4. Consider the generalized eigenvalue problem

$$\boldsymbol{A}\boldsymbol{v} = \lambda \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{v}, \quad \boldsymbol{A} = \begin{pmatrix} 3a & 1 \\ 1 & a \end{pmatrix}.$$

(a) Find the eigenvalues of $\boldsymbol{A}$, and explicitly determine the values of a for which $\mathrm{n}(\boldsymbol{A}) = 0$, $\mathrm{n}(\boldsymbol{A}) = 1$, and $\mathrm{n}(\boldsymbol{A}) = 2$. In particular, state the values of a for which $\mathrm{n}(\boldsymbol{A})$ changes.

(b) Find the eigenvalues for the generalized problem. Explicitly state the values of a for which the eigenvalues are purely real, and those values of a for which the eigenvalues have a nonzero imaginary part. For which values of a does the transition occur?

(c) Is there a relationship between the transition values found in part (a) and those found in part (b)? Explain.