Introduction to Ordinary Differential Equations

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0. Introduction

Some physically and/or biologically interesting mathematical models are:

(a) The Gross-Pitaevski equation,

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} - \phi + \phi^3 = V(t)\phi, \quad \lim_{|t|\to\infty} |\phi(t)| = 0,$$

is a model used in the study of Bose-Einstein condensates (see [9, 10, 13] and the references therein). Here $\phi(\cdot)$ represents the wavefunction of the condensate, and $V(\cdot)$ represents the applied external potential. The "boundary condition" guarantees that the condensate is localized, i.e., experimentally realizable.

(b) Lotka-Volterra competition model:

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = r_1N_1(1 - N_1/K_1) - b_1N_1N_2, \quad \frac{\mathrm{d}N_2}{\mathrm{d}t} = r_2N_2(1 - N_2/K_2) - b_2N_1N_2.$$

Here K_i represents the carrying capacity of the environment for species N_i in the absence of competition, and b_i reflects the competition between the two species.

(c) Firefly's flashing rhythm:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega + A\sin(\alpha - \theta), \quad \frac{\mathrm{d}\alpha}{\mathrm{d}t} = \Omega.$$

Here θ represents the phase of the firefly's rhythm, A is the firefly's ability to modify its frequency, and α is the periodic stimulus.

Different questions can be asked for each model. For example, when considering the Lotka-Volterra model, one can ask:

(a) does one species wipe out the other?

(b) if not, in which manner do the two species coexist - relatively constant populations, or populations which periodically fluctuate?

The purpose of this course is to acquire and develop the mathematical tools that will allow you to begin to analyze models such as the above. In the remainder of this section we will quickly review the material that you (should) have learned in your undergraduate course in Ordinary Differential Equations (e.g., see [1]), as well as your introductory course in real analysis (e.g., see [2]).

0.1. Notation and introductory definitions

Definition 0.1. A norm $|\cdot| : \mathbb{R}^n \mapsto \mathbb{R}$ satisfies

(a) $|\boldsymbol{x} + \boldsymbol{y}| \le |\boldsymbol{x}| + |\boldsymbol{y}|$

(b) $|c\boldsymbol{x}| = |c|\,|\boldsymbol{x}|$ for all $c \in \mathbb{R}$

(c) $|\boldsymbol{x}| \geq 0$, and equality occurs only if $\boldsymbol{x} = \boldsymbol{0}$

Definition 0.2. Let $\boldsymbol{x} = (x_1, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$. A norm is given by

$$|\boldsymbol{x}| := \sum_{i=1}^{n} |x_i|.$$

Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. The norm of $\boldsymbol{A}$ is given by

$$|\boldsymbol{A}| := \sup\{|\boldsymbol{A}\boldsymbol{x}| \,:\, |\boldsymbol{x}| = 1\} = \sum_{i,j=1}^{n} |a_{ij}|.$$

Remark 0.3. One has that:

(a) More generally, a norm can be defined by

$$|\boldsymbol{x}|_p := (\sum_{i=1}^{n} |x_i|^p)^{1/p}.$$

It can be shown that each of these norms are equivalent, i.e., given a $1 \leq p, q \leq \infty$, one has that there are positive constants c_1 and c_2 such that

$$c_1|\boldsymbol{x}|_q \leq |\boldsymbol{x}|_p \leq c_2|\boldsymbol{x}|_q.$$

For example,

$$|\boldsymbol{x}|_2 \leq |\boldsymbol{x}|_1 \leq \sqrt{n}|\boldsymbol{x}|_2.$$

Thus, the choice of the norm is not important.

(b) $|\boldsymbol{A}\boldsymbol{x}| \leq |\boldsymbol{A}|\,|\boldsymbol{x}|$.

Definition 0.4. Given an $\boldsymbol{x}_0 \in \mathbb{R}^n$ and $\gamma > 0$, define

$$B(\boldsymbol{x}_0, \gamma) := \{\boldsymbol{x} \in \mathbb{R}^n \,:\, |\boldsymbol{x} - \boldsymbol{x}_0| < \gamma\}, \quad \overline{B(\boldsymbol{x}_0, \gamma)} := \{\boldsymbol{x} \in \mathbb{R}^n \,:\, |\boldsymbol{x} - \boldsymbol{x}_0| \leq \gamma\}.$$

Regarding calculus for vectors, we write:

(a) $\int \boldsymbol{x}(t)\,\mathrm{d}t = (\int x_1(t)\,\mathrm{d}t, \dots, \int x_n(t)\,\mathrm{d}t)^{\mathrm{T}}$

(b) $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = (\mathrm{d}x_1/\mathrm{d}t, \dots, \mathrm{d}x_n/\mathrm{d}t)^{\mathrm{T}}$

Definition 0.5. Let $G \subset \mathbb{R} \times \mathbb{R}^n$ be open, and let $\boldsymbol{f} \,:\, G \mapsto \mathbb{R}^n$ be continuously differentiable. The matrix $\mathrm{D}\boldsymbol{f} := \partial\boldsymbol{f}/\partial\boldsymbol{x} \in \mathbb{R}^{n\times n}$ satisfies

$$(\mathrm{D}\boldsymbol{f})_{ij} = \frac{\partial f_i}{\partial x_j}.$$

Definition 0.6. Let $\boldsymbol{f} \,:\, G \mapsto \mathbb{R}^n$ be continuous. An ordinary differential equation (ODE) is of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \quad \dot{} := \frac{\mathrm{d}}{\mathrm{d}t}.$$

The function $\boldsymbol{x} = \phi(t)$ solves the ODE on an open interval $I \subset \mathbb{R}$ if $\phi \,:\, I \mapsto \mathbb{R}^n$ is continuously differentiable with $\dot{\phi} = \boldsymbol{f}(t, \phi)$.

Remark 0.7. Consider the second-order scalar equation

$$\ddot{y} + y - y^2 = \sin t.$$

Upon setting $x_1 := y, x_2 := \dot{y}$, one gets the first-order system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_1^2 + \sin t,$$

i.e.,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}),$$

where

$$\boldsymbol{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \boldsymbol{f}(t, \boldsymbol{x}) := \begin{pmatrix} x_2 \\ -x_1 + x_1^2 + \sin t \end{pmatrix}.$$

This trick can be used to transform a scalar equation of order n to a first-order system with n equations.

0.2. Solving linear systems

Now let us refresh our memories as to how one can explicitly solve linear ODEs of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \tag{0.1}$$

where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. Substituting

$$\boldsymbol{x} := \mathrm{e}^{\lambda t}\boldsymbol{v} \tag{0.2}$$

into equation (0.1) yields

$$(A - \lambda\mathbb{1})\boldsymbol{v} = \boldsymbol{0}.$$

In the above the vector $\boldsymbol{v}$ is known as the eigenvector, and the corresponding eigenvalue λ is found by solving the characteristic equation

$$\det(\boldsymbol{A} - \lambda\mathbb{1}) = 0.$$

If $\lambda \in \mathbb{R}$, then the solution with real-valued components is given in equation (0.2). If $\lambda \in \mathbb{C}$, i.e., $\lambda = a + \mathrm{i}b$, then the corresponding eigenvector is given by $\boldsymbol{v} = \boldsymbol{p} + \mathrm{i}\boldsymbol{q}$, where $\boldsymbol{v}, \boldsymbol{q} \in \mathbb{R}^n$, and the two linearly independent solutions with real-valued components are given by

$$\boldsymbol{x}_1 = \mathrm{e}^{at}\left(\cos(bt)\boldsymbol{p} - \sin(bt)\boldsymbol{q}\right), \quad \boldsymbol{x}_2 = \mathrm{e}^{at}\left(\sin(bt)\boldsymbol{p} + \cos(bt)\boldsymbol{q}\right).$$

If the eigenvalues are simple, then one can find n linearly independent solutions $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ via the manner proscribed above, and the general solution is then given by

$$\boldsymbol{x} = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n,$$

where $c_j \in \mathbb{R}$ for $j = 1, \dots, n$.

0.3. The phase plane for linear systems

Now suppose that $n = 2$. The eigenvalues are zeros of the characteristic equation

$$\lambda^2 - \mathrm{trace}(\boldsymbol{A})\lambda + \det(\boldsymbol{A}) = 0,$$

i.e.,

$$\lambda = \lambda_\pm := \frac{1}{2}\left(\mathrm{trace}(\boldsymbol{A}) \pm \sqrt{\mathrm{trace}(\boldsymbol{A})^2 - 4\det(\boldsymbol{A})}\right).$$

0.3.1. Real eigenvalues

First suppose that $\mathrm{trace}(\boldsymbol{A})^2 > 4\det(\boldsymbol{A})$, so that $\lambda_- < \lambda_+ \in \mathbb{R}$. When graphing trajectories, we will use the fact that the line in the xy-plane through the origin parallel to the vector

$$\boldsymbol{v} = \begin{pmatrix} c \\ d \end{pmatrix}$$

is given by

$$y = \frac{d}{c}x.$$

If $\det(\boldsymbol{A}) < 0$, then $\lambda_- < 0 < \lambda_+$, and the critical point $\boldsymbol{x} = \boldsymbol{0}$ is known as an unstable saddle point. For example, suppose that

$$\boldsymbol{A} = \begin{pmatrix} 3 & 5 \\ -2 & -4 \end{pmatrix}. \tag{0.3}$$

The eigenvalues and associated eigenvectors are given by

$$\lambda_- = -2,\ \boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}; \quad \lambda_+ = 1,\ \boldsymbol{v}_2 = \begin{pmatrix} -5 \\ 2 \end{pmatrix},$$

so that the general solution is given by

$$\boldsymbol{x}(t) = c_1 \mathrm{e}^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathrm{e}^{t} \begin{pmatrix} -5 \\ 2 \end{pmatrix}.$$

When $c_2 = 0$ the solutions are restricted to the line $y = -x$; furthermore, any solution on this line goes to the origin exponentially fast. When $c_1 = 0$ the solutions are restricted to the line $y = -2/5\,x$; furthermore, any solution on this line goes grows large exponentially fast. Sample trajectories are given below:

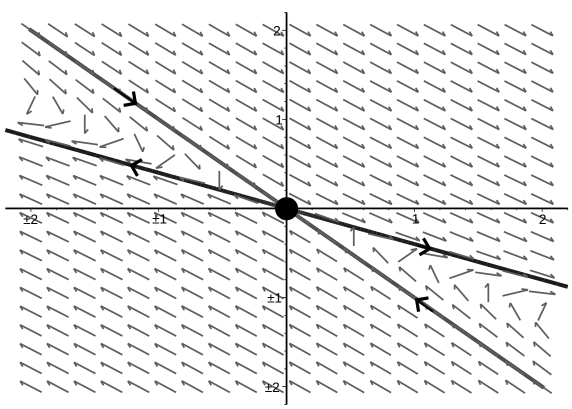

Figure 1: The phase portrait for equation (0.3).

If $\det(\boldsymbol{A}) > 0$, then $\mathrm{sign}(\lambda_\pm) = \mathrm{sign}(\mathrm{trace}(\boldsymbol{A}))$, which implies that if $\mathrm{trace}(\boldsymbol{A}) < 0$, then $\boldsymbol{x} = \boldsymbol{0}$ is a stable node. For example, suppose that

$$\boldsymbol{A} = \begin{pmatrix} 3 & 5 \\ -4 & -6 \end{pmatrix}. \tag{0.4}$$

The eigenvalues and associated eigenvectors are given by

$$\lambda_- = -2,\ \boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}; \quad \lambda_+ = -1,\ \boldsymbol{v}_2 = \begin{pmatrix} -5 \\ 4 \end{pmatrix},$$

so that the general solution is given by

$$\boldsymbol{x}(t) = c_1 \mathrm{e}^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathrm{e}^{-t} \begin{pmatrix} -5 \\ 4 \end{pmatrix}.$$

When $c_2 = 0$ the solutions are restricted to the line $y = -x$; furthermore, any solution on this line goes to the origin exponentially fast. When $c_1 = 0$ the solutions are restricted to the line $y = -4/5\,x$; furthermore, any solution on this line also goes to the origin exponentially fast. Sample trajectories are given below:

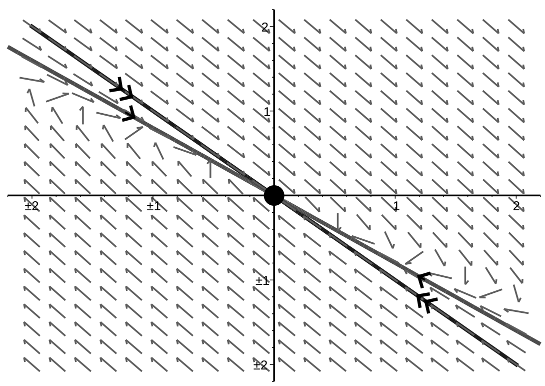

Figure 2: The phase portrait for equation (0.4).

On the other hand, if $\operatorname{trace}(\boldsymbol{A}) > 0$, then $\boldsymbol{x} = \boldsymbol{0}$ is an unstable node. For example, suppose that

$$\boldsymbol{A} = \begin{pmatrix} -3 & 5 \\ -4 & 6 \end{pmatrix}. \tag{0.5}$$

The eigenvalues and associated eigenvectors are given by

$$\lambda_- = 1,\ \boldsymbol{v}_2 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}; \quad \lambda_+ = 2,\ \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so that the general solution is given by

$$\boldsymbol{x}(t) = c_1 \mathrm{e}^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \mathrm{e}^{-t} \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

When $c_2 = 0$ the solutions are restricted to the line $y = x$; furthermore, any solution on this line grows large exponentially fast. When $c_1 = 0$ the solutions are restricted to the line $y = 4/5\,x$; furthermore, any solution on this line also grows large exponentially fast. Sample trajectories are given below:

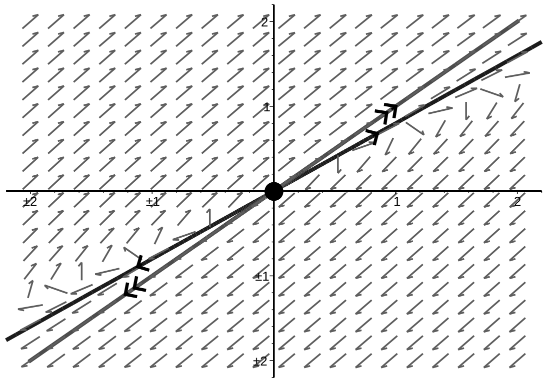

Figure 3: The phase portrait for equation (0.5).

0.3.2. Complex eigenvalues

If $\mathrm{trace}(\boldsymbol{A})^2 < 4\det(\boldsymbol{A})$, i.e.,

$$\lambda_\pm = \alpha \pm \mathrm{i}\beta \in \mathbb{C}, \quad \beta \in \mathbb{R}^+,$$

then it turns out to be the case that through a clever change of variables the system in equation (0.1) is equivalent to

$$\dot{\boldsymbol{y}} = \boldsymbol{B}\boldsymbol{y}, \quad \boldsymbol{B} := \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}. \tag{0.6}$$

We will now study equation (0.6) in coordinates, i.e.,

$$\begin{aligned} \dot{y}_1 &= \alpha y_1 - \beta y_2 \\ \dot{y}_2 &= \beta y_1 + \alpha y_2. \end{aligned}$$

Upon using polar coordinates, i.e., setting

$$y_1 = r\cos\theta, \quad y_2 = r\sin\theta.$$

and using implicit differentiation,

$$\dot{y}_1 = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \quad \dot{y}_2 = \dot{r}\sin\theta + r\dot{\theta}\cos\theta,$$

it is seen that

$$\dot{r} = \dot{y}_1\cos\theta + \dot{y}_2\sin\theta, \quad r\dot{\theta} = -\dot{y}_1\sin\theta + \dot{y}_2\cos\theta.$$

Simplifying the above yields

$$\dot{r} = \alpha r, \quad \dot{\theta} = \beta,$$

i.e.,

$$r(t) = r(0)\mathrm{e}^{\alpha t}, \quad \theta(t) = \beta t + \theta(0).$$

Thus, the solution to equation (0.6) is given by

$$y_1(t) = r(0)\mathrm{e}^{\alpha t}\cos(\beta t + \theta(0)), \quad y_2(t) = r(0)\mathrm{e}^{\alpha t}\sin(\beta t + \theta(0)),$$

where

$$r(0) = \sqrt{y_1(0)^2 + y_2(0)^2}, \quad \tan\theta(0) = \frac{y_2(0)}{y_1(0)}.$$

The form of the solution guarantees that the trajectories will spiral about the origin. If $\alpha > 0$, i.e., $\mathrm{trace}(\boldsymbol{A}) > 0$, then the solutions will spiral away from the origin exponentially fast; in this case, the origin is an unstable spiral point. A sample trajectory is given below:

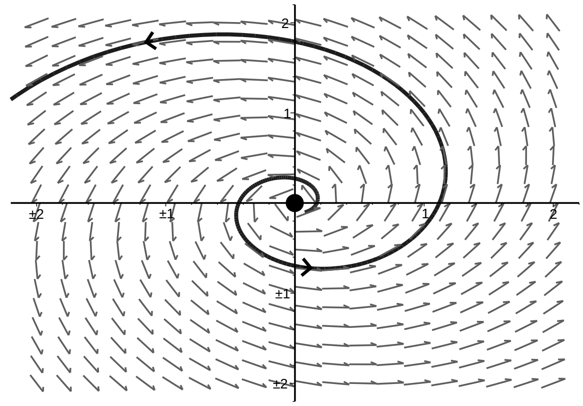

Figure 4: The origin is an unstable spiral point.

If $\alpha < 0$, i.e., trace($\boldsymbol{A}$) < 0, the solutions will spiral towards the origin exponentially fast; in this case, the origin is a stable spiral point. A sample trajectory is given below:

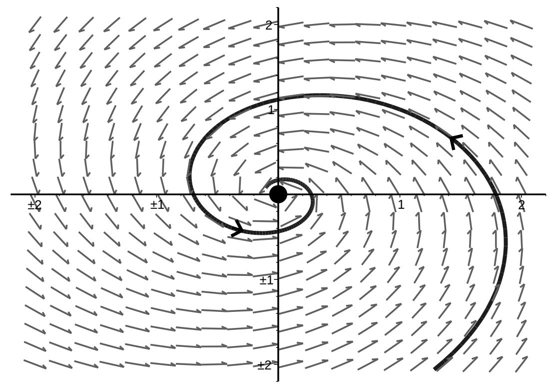

Figure 5: The origin is a stable spiral point.

Finally, if $\alpha = 0$, i.e., trace($\boldsymbol{A}$) $= 0$, the trajectories will be closed; in this case, the origin is a center. A sample trajectory is given below:

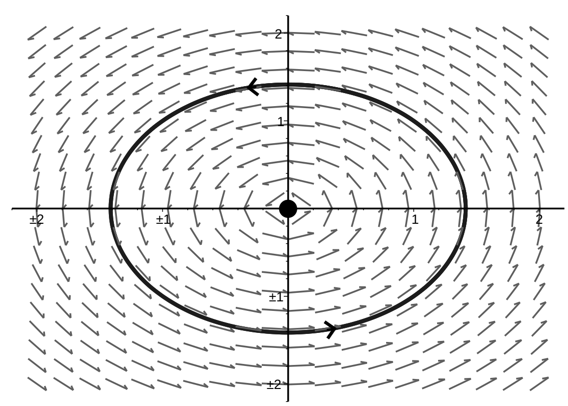

Figure 6: The origin is a center.

The last item to be determined is the direction of spiralling - is it counterclockwise or clockwise? The answer, of course, depends on the particular problem. One strategy which will help in answering the question is illustrated in the following example. Suppose that

$$\boldsymbol{A} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}.$$

The eigenvalues are $\lambda_\pm = -1 \pm 2\mathrm{i}$, so that the origin is a stable spiral point. Write the system as

$$\begin{aligned} \dot{x} &= -x - 4y \\ \dot{y} &= x - y. \end{aligned} \tag{0.7}$$

On the half-line $x = 0$ and $y > 0$ the vector field satisfies

$$\dot{x} = -4y < 0.$$

As a consequence, $x(t)$ is decreasing whenever the trajectory hits this line, which implies that the motion is counterclockwise. A sample trajectory is given below.

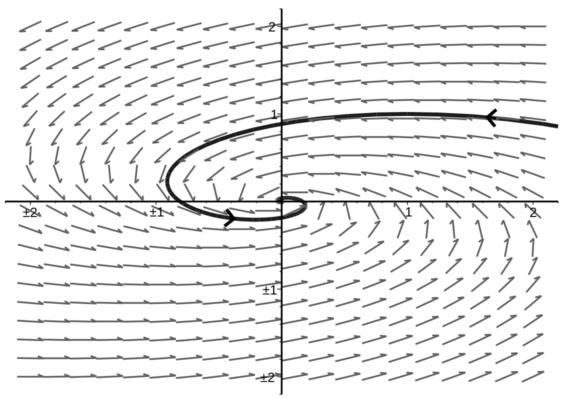

Figure 7: The phase portrait for equation (0.7).

Remark 0.8. If $\dot{x} > 0$ on the half-line $x = 0$ and $y > 0$, then the motion will be clockwise.

0.3.3. Classification of the critical point

As it can be seen in the examples, the stability of the critical point depends upon the sign of the eigenvalues. The following table summarizes the above discussion:

Eigenvalues	Type of Critical Point
$\lambda_1, \lambda_2 > 0$	Unstable node
$\lambda_1, \lambda_2 < 0$	Stable node
$\lambda_1 < 0 < \lambda_2$	Unstable saddle point
$\lambda = \alpha \pm \mathrm{i}\beta$, $\alpha \in \mathbb{R}^+$	Unstable spiral point
$\lambda = \alpha \pm \mathrm{i}\beta$, $\alpha \in \mathbb{R}^-$	Stable spiral point
$\lambda = \pm\mathrm{i}\beta$	Linear center

0.4. The phase plane for conservative nonlinear systems

Finally, let us briefly discuss phase portraits for planar vector fields, i.e., the graphical representation of solution curves to

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^2. \tag{0.8}$$

For equation (0.8) suppose that $E : \mathbb{R}^2 \mapsto \mathbb{R}$ is a first integral, i.e., along trajectories

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \nabla E(\boldsymbol{x}) \cdot \dot{\boldsymbol{x}} = 0.$$

One then has that $E(\boldsymbol{x}(t)) = E(\boldsymbol{x}_0)$ for all $t \in \mathbb{R}$; hence, solutions reside on level curves, and the set of all level curves gives all of the trajectories.

For example, if

$$\boldsymbol{f}(\boldsymbol{x}) = (x_2, -g(x_1))^{\mathrm{T}},$$

which arises when equation (0.8) is equivalent to the second-order problem

$$\ddot{x} + g(x) = 0,$$

then

$$E(\boldsymbol{x}) = \frac{1}{2}x_2^2 + \int_0^{x_1} g(s)\,\mathrm{d}s$$

is a first integral. Here the functional E represents the total energy for the conservative physical system. When considering the pendulum equation, i.e., $g(x) = \sin x$, one has that

$$E(\boldsymbol{x}) = \frac{1}{2}x_2^2 + \int_0^{x_1} \sin s\,\mathrm{d}s.$$

In this case if $E(\boldsymbol{x}) < 2$, then the solution is periodic.

1. Existence and uniqueness

Consider the initial value problem (IVP)

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \tag{1.1}$$

where $\boldsymbol{f}$ is continuous on an open set G with $(t_0, \boldsymbol{x}_0) \in G$. When considering the nonautonomous equation (1.1), one can make it autonomous by rewriting it as

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \boldsymbol{f}(t, \boldsymbol{x}), \quad \frac{\mathrm{d}t}{\mathrm{d}s} = 1; \quad (\boldsymbol{x}(0), t(0)) = (\boldsymbol{x}_0, t_0),$$

i.e.,

$$\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y}), \quad \boldsymbol{y}(0) = \boldsymbol{y}_0; \quad \boldsymbol{y} = (\boldsymbol{x}, t), \quad \boldsymbol{g}(y) = (\boldsymbol{f}(t, \boldsymbol{x}), 1).$$

Hence, without loss of generality one can consider the IVP

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{1.2}$$

where $\boldsymbol{f} : G \subset \mathbb{R}^n \mapsto \mathbb{R}^n$ is continuous. From this point forward, it will be assumed that the ODE under consideration is autonomous and given by equation (1.2).

1.1. Existence

Theorem 1.1 (Peano's Existence Theorem)**.** *Let $\boldsymbol{x}_0 \in G$ be given. There is a $\delta > 0$ and a function $\boldsymbol{x}(t)$ defined on $I = (-\delta, \delta)$ which solves equation (1.2).*

Remark 1.2. The solution may not be unique; for example, consider $\dot{x} = \sqrt{|x|}$, $x(0) = 0$, which has the family of solutions $x(t) = x_a(t)$, where $a \in \mathbb{R}^+$ and

$$x_a(t) := \begin{cases} 0, & 0 \le t \le a \\ \frac{1}{4}(t-a)^2, & a \le t. \end{cases}$$

1.1.1. Proof by successive approximations

Herein an additional assumption on $\boldsymbol{f}$ will be made; precisely, that $\boldsymbol{f}$ is uniformly Lipschitz continuous on G (see Definition 1.14). This is for technical reasons only, and as it is seen in Section 1.1.2, it can be removed with a different method of proof of Theorem 1.1.

Let $\gamma > 0$ be chosen so that $\overline{B(\boldsymbol{x}_0, \gamma)} \subset G$. Since $\boldsymbol{f}$ is continuous, there is a $C > 0$ such that $|\boldsymbol{f}(\boldsymbol{x})| < C$ for $\boldsymbol{x} \in \overline{B(\boldsymbol{x}_0, \gamma)}$. Set $\boldsymbol{x}_0(t) \equiv \boldsymbol{x}_0$, and for a given $n \in \mathbb{N}$ and for $j = 1, \dots, n$ define the sequence $\{\boldsymbol{x}_j(t)\}$ via

$$\boldsymbol{x}_{j+1}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{f}(\boldsymbol{x}_j(s))\,\mathrm{d}s. \tag{1.3}$$

Note that $\boldsymbol{x}_j(0) \in B(\boldsymbol{x}_0, \gamma)$ for each j. Since $\boldsymbol{f}$ is uniformly bounded, one has that

$$|\boldsymbol{x}_{j+1}(t) - \boldsymbol{x}_0| \le \int_0^{|t|} |\boldsymbol{f}(\boldsymbol{x}_j(s))|\,\mathrm{d}s < C|t|; \tag{1.4}$$

thus, for $|t| \le \delta := \gamma/C$ one has that $\boldsymbol{x}_j(t) \in \overline{B(\boldsymbol{x}_0, \gamma)}$. Furthermore, since $\boldsymbol{f}$ is continuous one has that each $\boldsymbol{x}_j(t)$ is continuous on $[-\delta, \delta]$.

It will now be shown by induction that

$$|\boldsymbol{x}_{j+1}(t) - \boldsymbol{x}_j(t)| \le \frac{CK^j|t|^{j+1}}{(j+1)!}, \quad j \in \mathbb{N}_0, \tag{1.5}$$

where K is the Lipschitz constant for $\boldsymbol{f}$. Now, by equation (1.4) one has that equation (1.5) holds for $j = 0$. Assume that it holds for $j = 0, \dots, n$. By equation (1.3) one has that for $n \geq 1$,

$$\boldsymbol{x}_{n+1}(t) - \boldsymbol{x}_n(t) = \int_0^t [\boldsymbol{f}(\boldsymbol{x}_n(s)) - \boldsymbol{f}(\boldsymbol{x}_{n-1}(s))] \, \mathrm{d}s.$$

Since $\boldsymbol{f}$ is Lipschitz this then implies that

$$|\boldsymbol{x}_{n+1}(t) - \boldsymbol{x}_n(t)| \leq K \int_0^{|t|} |\boldsymbol{x}_n(s)) - \boldsymbol{x}_{n-1}(s)| \, \mathrm{d}s,$$

which by equation (1.5) with $j = n - 1$ further yields that

$$|\boldsymbol{x}_{n+1}(t) - \boldsymbol{x}_n(t)| \leq \frac{CK^n}{n!} \int_0^{|t|} s^n \, \mathrm{d}s.$$

Hence, equation (1.5) holds.

Now set

$$\boldsymbol{x}(t) := \boldsymbol{x}_0 + \sum_{j=0}^{+\infty} [\boldsymbol{x}_{j+1}(t) - \boldsymbol{x}_j(t)].$$

As a consequence of equation (1.5) the infinite sum is uniformly convergent, and since each term is continuous, this then implies that $\boldsymbol{x}(t)$ is continuous on $[-\delta, \delta]$. Since the sum is telescoping, one actually has that

$$\boldsymbol{x}(t) = \lim_{j \to +\infty} \boldsymbol{x}_j(t),$$

with the limit being uniform on $[-\delta, \delta]$. Since $\boldsymbol{f}$ is uniformly Lipschitz continuous one has that $\boldsymbol{f}(\boldsymbol{x}_j(t)) \to \boldsymbol{f}(\boldsymbol{x}(t))$ uniformly on $[-\delta, \delta]$. Hence, from equation (1.3) one can conclude that

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{f}(\boldsymbol{x}(s)) \, \mathrm{d}s, \quad -\delta \leq t \leq \delta. \tag{1.6}$$

The right-hand side of equation (1.6) is differentiable on $(-\delta, \delta)$; hence, upon differentiating one has that $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. The solution $\boldsymbol{x}$ also clearly satisfies $\boldsymbol{x}(0) = \boldsymbol{x}_0$. In conclusion, one has that the solution to equation (1.6) satisfies equation (1.2). Note that as a consequence of the discussion leading to equation (1.6), one has the following result.

Corollary 1.3. *$\boldsymbol{x}(t)$ solves equation (1.2) if and only if $\boldsymbol{x}(t)$ solves the integral equation equation (1.6).*

1.1.2. Proof by polygonal approximations

Euler's method (a numerical method of $\mathcal{O}(h)$) is given by

$$\boldsymbol{x}_{n+1}(t) = \boldsymbol{x}_n(t_n) + (t - t_n)\boldsymbol{f}(\boldsymbol{x}_n), \quad t_{n+1} = t_n + h, \quad t \in [t_n, t_{n+1}].$$

One can define a continuous piecewise linear function via

$$\boldsymbol{x}(t; h) = \boldsymbol{x}_k(t), \quad t \in [t_k, t_{k+1}].$$

The goal herein is to show that under the appropriate assumptions on the vector field Euler's method converges to a solution of equation (1.1), i.e., that there exists a sequence $\{h_j\}$ with $h_j \to 0^+$ as $j \to +\infty$ such that $\lim_{j \to +\infty} \boldsymbol{x}(t; h_j)$ is a solution to equation (1.1).

Definition 1.4. Let $\{\boldsymbol{x}_k(t)\}_{k \in \mathbb{N}}$ be a family of functions defined on $I := (a, b)$. The sequence converges uniformly to $\boldsymbol{x}(t)$ if for every $\epsilon > 0$ there is an $N(\epsilon)$ such that $|\boldsymbol{x}_k(t) - \boldsymbol{x}(t)| < \epsilon$ for $k > N$ and $t \in I$.

Proposition 1.5. *If the functions $\boldsymbol{x}_k(t)$ are continuous, then $\boldsymbol{x}(t)$ is continuous.*

Definition 1.6. The sequence $\{\boldsymbol{x}_k(t)\}$ is equicontinuous on I if for every $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that $|\boldsymbol{x}_k(t) - \boldsymbol{x}_k(s)| < \epsilon$ if $|t - s| < \delta$ for all $s, t \in I$ and $k \in \mathbb{N}$.

Definition 1.7. The sequence $\{\boldsymbol{x}_k(t)\}$ is uniformly bounded if $|\boldsymbol{x}_k(t)| < M$ for all $t \in I$ and $k \in \mathbb{N}$.

Theorem 1.8 (Arzela-Ascoli Theorem)**.** *If $\{\boldsymbol{x}_k(t)\}$ is an equicontinuous, uniformly bounded sequence of functions on I, then there is a subsequence $\{\boldsymbol{x}_{k_j}(t)\}$ which converges uniformly to $\boldsymbol{x}(t)$ on I.*

Remark 1.9. It is not necessarily true that $\boldsymbol{x}_k(t) \to \boldsymbol{x}(t)$ as $k \to +\infty$; furthermore, different subsequences may converge to different functions.

Proof: The proof will take place in 6 steps.

I. For $\delta > 0$ given, subdivide $[0, \delta]$ into n equal subintervals $0 = t_0 < t_1 < \cdots < t_n = \delta$, so that $t_{k+1} - t_k = \delta/n := h_n$. Set

$$\boldsymbol{x}_{k+1}(t) = \boldsymbol{x}_k(t_k) + (t - t_k)\boldsymbol{f}(\boldsymbol{x}_k(t_k)), \quad t \in [t_k, t_{k+1}],$$

and define $\boldsymbol{x}(t; h_n) = \boldsymbol{x}_{k+1}(t)$ for $t \in [t_k, t_{k+1}]$. Note that $\boldsymbol{x}(t; h_n)$ is piecewise linear and continuous. In all that follows, for $k = 0, \dots, n-1$ define $\boldsymbol{x}_k := \boldsymbol{x}(t_k; h_n)$.

It must now be shown that by making δ sufficiently small, $\boldsymbol{x}(t; h_n) \in G$ for all n. Let $\gamma > 0$ be chosen so that $\overline{B(\boldsymbol{x}_0, \gamma)} \subset G$. Since $\boldsymbol{f}$ is continuous, there is a $C > 0$ such that $|\boldsymbol{f}(\boldsymbol{x})| < C$ for $\boldsymbol{x} \in \overline{B(\boldsymbol{x}_0, \gamma)}$. Thus, for $t \in [0, t_1]$,

$$|\boldsymbol{x}(t; h_n) - \boldsymbol{x}_0| = t|\boldsymbol{f}(\boldsymbol{x}_0)| < Ch_n = \frac{C\delta}{n}.$$

Assuming that $C\delta < \gamma$ (the smallness condition on δ), one then has that $|\boldsymbol{x}(t_1; h_n) - \boldsymbol{x}_0| < \gamma/n$, so that $\boldsymbol{x}_1 \in \overline{B(\boldsymbol{x}_0, \gamma)}$. Similarly, for $t \in [t_1, t_2]$ one has

$$|\boldsymbol{x}(t; h_n) - \boldsymbol{x}_1| < (t - t_1)C \leq Ch_n < \frac{\gamma}{n},$$

so that $\boldsymbol{x}_2$ satisfies $|\boldsymbol{x}_2 - \boldsymbol{x}_1| < \gamma/n$. Thus,

$$|\boldsymbol{x}_2 - \boldsymbol{x}_0| \leq |\boldsymbol{x}_2 - \boldsymbol{x}_1| + |\boldsymbol{x}_1 - \boldsymbol{x}_0| < \frac{2\gamma}{n},$$

i.e., $\boldsymbol{x}_2 \in \overline{B(\boldsymbol{x}_0, \gamma)}$. Continuing in this manner it is seen that for $t \in [t_j, t_{j+1}]$, $j = 0, \dots, n-1$,

$$|\boldsymbol{x}(t; h_n) - \boldsymbol{x}_0| < \frac{(j+1)\gamma}{n} \leq \gamma,$$

so that $\boldsymbol{x}(t; h_n) \in \overline{B(\boldsymbol{x}_0, \gamma)}$ for all $t \in [0, \delta]$.

II. Since $|\boldsymbol{x}(t; h_n) - \boldsymbol{x}_0| < \gamma$ for $t \in [0, \delta]$, one has that $|\boldsymbol{x}(t; h_n)| < |\boldsymbol{x}_0| + \gamma$. Hence, the sequence $\{\boldsymbol{x}(t; h_n)\}$ is uniformly bounded.

III. It must now be shown that $\{\boldsymbol{x}(t; h_n)\}$ is equicontinuous, so we need to estimate $|\boldsymbol{x}(t; h_n) - \boldsymbol{x}(s; h_n)|$. Assume that $s < t$, and further assume that they do not occupy the same subinterval. There is an i, j such that

$$t_{i-1} < s \leq t_i < t_{i+1} < \cdots < t_j \leq t < t_{j+1}.$$

Now,

$$\boldsymbol{x}(t; h_n) = \boldsymbol{x}_j + (t - t_j)\boldsymbol{f}(\boldsymbol{x}_j), \quad \boldsymbol{x}(s; h_n) = \boldsymbol{x}_{i-1} + (s - t_{i-1})\boldsymbol{f}(\boldsymbol{x}_{i-1}),$$

and since $\boldsymbol{x}_i = \boldsymbol{x}_{i-1} + (t_i - t_{i-1})\boldsymbol{f}(\boldsymbol{x}_{i-1})$, one has that

$$\boldsymbol{x}(s; h_n) = \boldsymbol{x}_i + (s - t_i)\boldsymbol{f}(\boldsymbol{x}_{i-1}).$$

Now,

$$\boldsymbol{x}(t; h_n) - \boldsymbol{x}(s; h_n) = \boldsymbol{x}(t; h_n) - \boldsymbol{x}_j + (\boldsymbol{x}_j - \boldsymbol{x}_{j-1}) + \cdots + (\boldsymbol{x}_{i+1} - \boldsymbol{x}_i) + \boldsymbol{x}_i - \boldsymbol{x}(s; h_n),$$

so the identity $\boldsymbol{x}_{k+1} - \boldsymbol{x}_k = h_n f(\boldsymbol{x}_k)$ yields

$$\boldsymbol{x}(t; h_n) - \boldsymbol{x}(s; h_n) = (t - t_j)\boldsymbol{f}(\boldsymbol{x}_j) + (t_i - s)\boldsymbol{f}(\boldsymbol{x}_{i-1}) + h_n \sum_{k=i}^{j-1} \boldsymbol{f}(\boldsymbol{x}_k). \tag{1.7}$$

Since $|\boldsymbol{f}(\boldsymbol{x})| < C$ for $\boldsymbol{x} \in \overline{B(\boldsymbol{x}_0, \gamma)}$, this yields

$$\begin{aligned}|\boldsymbol{x}(t; h_n) - \boldsymbol{x}(s; h_n)| &< [(t - t_j) + (t_i - s) + (j - i)h_n]C \\ &< C(t - s).\end{aligned}$$

IV. By the Arzela-Ascoli Theorem there is a subsequence $\{\boldsymbol{x}(t; h_{n_k})\}$ which is continuous and converges to a continuous $\boldsymbol{x}(t)$ for $t \in [0, \delta]$. Furthermore, since $\boldsymbol{x}(0; h_n) = \boldsymbol{x}_0$ for all n, one has that $\boldsymbol{x}(0) = \boldsymbol{x}_0$.

V. Without loss of generality, suppose that the full sequence converges to $\boldsymbol{x}(t)$. A careful examination of the sequence reveals that for $t \in [t_k, t_{k+1})$,

$$\boldsymbol{x}(t; h_n) = \boldsymbol{x}_0 + (t - t_k)\boldsymbol{f}(\boldsymbol{x}_k) + h_n \sum_{j=0}^{k-1} \boldsymbol{f}(\boldsymbol{x}_j),$$

where $\boldsymbol{x}_j = \boldsymbol{x}(t_j; h_n)$. Since the sequence converges uniformly to $\boldsymbol{x}(t)$, and since $\boldsymbol{f}$ is continuous, upon taking the limit it is seen that $\boldsymbol{x}(t)$ solves the integral equation

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s. \tag{1.8}$$

Since $\boldsymbol{f}(\boldsymbol{x}(t))$ is continuous, the right-hand side is differentiable; hence, $\boldsymbol{x}(t)$ is differentiable. Taking the derivative yields

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

i.e., $\boldsymbol{x}$ solves equation (1.2).

VI. We need to get a solution on $[-\delta, \delta]$. Set $s = -t$, and consider

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}s} = -\boldsymbol{f}(\boldsymbol{y}(s)).$$

If a solution $\boldsymbol{y}(s)$ exists, then $\boldsymbol{x}(t) = \boldsymbol{y}(-s)$ is a solution to the original ODE, as

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = -\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}s} = \boldsymbol{f}(\boldsymbol{y}(s)) = \boldsymbol{f}(\boldsymbol{y}(-t)) = \boldsymbol{f}(\boldsymbol{x}(t)).$$

Thus, by the previous steps we have a continuous solution defined on $[-\delta, \delta]$, and with it being differentiable on $(-\delta, \delta)\backslash\{0\}$. We need to show that $\dot{\boldsymbol{x}}(0) = \boldsymbol{f}(\boldsymbol{x}_0)$. The argument in **V.** can be used to show that $\boldsymbol{x}(t)$ is differentiable from the right at $t = 0$, with the right-hand derivative being $\boldsymbol{f}(\boldsymbol{x}_0)$. Similarly, the left-hand derivative will be $\boldsymbol{f}(\boldsymbol{x}_0)$. The proof is now complete. □

Remark 1.10. As a consequence of the discussion leading to equation (1.8), one again has Corollary 1.3.

1.2. Uniqueness

Now that it is known that equation (1.2) has a solution (given in equation (1.6)), it is necessary to determine the conditions under which it is unique. Afterwards, we must then understand how the maximal interval of existence can be determined. The following lemma will be crucial in answering these questions.

Lemma 1.11 (Gronwall's inequality)**.** *Suppose that $a < b$, and let α, β, ψ be nonnegative continuous functions defined on $[a, b]$. Furthermore, suppose that either α is constant, or α is differential on (a, b) with $\dot{\alpha} > 0$. If*

$$\beta(t) \le \alpha(t) + \int_a^t \psi(s)\beta(s)\,\mathrm{d}s, \quad t \in [a, b],$$

then

$$\beta(t) \le \alpha(t)\mathrm{e}^{\int_a^t \psi(s)\,\mathrm{d}s}, \quad t \in [a, b].$$

Proof: See [**17**, Theorem 1.1.2] or [**8**, Theorem III.1.1]. □

Theorem 1.12 (Taylor's theorem)**.** *Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open. If $\boldsymbol{f} : U \mapsto V$ is C^1 and $\boldsymbol{x} + t\boldsymbol{h} \in U$ for all $t \in [0, 1]$, then*

$$\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) = \left(\int_0^1 \mathrm{D}\boldsymbol{f}(\boldsymbol{x}+t\boldsymbol{h})\,\mathrm{d}t \right) \boldsymbol{h}.$$

Corollary 1.13. *If $\boldsymbol{f} : U \mapsto V$ is C^1 and $\boldsymbol{x} + t\boldsymbol{h} \in U$ for all $t \in [0, 1]$, then one has the estimate*

$$|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x})| \le C|\boldsymbol{h}|,$$

where

$$C := \sup_{t\in[0,1]} |\mathrm{D}f(\boldsymbol{x}+t\boldsymbol{h})|.$$

Definition 1.14. $\boldsymbol{f} : U \mapsto V$ is Lipschitz if there exists an $L > 0$ such that $|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})| \le L|\boldsymbol{x} - \boldsymbol{y}|$ for all $\boldsymbol{x}, \boldsymbol{y} \in U$. $\boldsymbol{f}$ is locally Lipschitz if for each $\boldsymbol{x}_0 \in U$ and each $\epsilon > 0$ such that $B(\boldsymbol{x}_0, \epsilon) \subset U$ there is an L_ϵ such that if $\boldsymbol{x}, \boldsymbol{y} \in B(\boldsymbol{x}_0, \epsilon)$, then $|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})| \le L_\epsilon|\boldsymbol{x} - \boldsymbol{y}|$.

As a consequence of Corollary 1.13 one has the following result.

Proposition 1.15. *If $\boldsymbol{f} : U \mapsto V$ is C^1, then f is locally Lipschitz.*

It is clear that if $\boldsymbol{f}$ is locally Lipschitz, then $\boldsymbol{f}$ is continuous. Hence, as a consequence of Theorem 1.1 if $\boldsymbol{f}$ is locally Lipschitz there exists a solution to equation (1.2). As it will be seen below, the mild restriction of $\boldsymbol{f}$ being locally Lipschitz, instead of merely being continuous, actually yields more than simple existence.

Theorem 1.16 (Uniqueness theorem)**.** *If $\boldsymbol{f}$ is locally Lipschitz, then the solution to equation (1.2) is unique.*

Proof: Since $\boldsymbol{f}$ is locally Lipschitz, $\boldsymbol{f}$ is continuous; hence, by Peano's Theorem 1.1 there is a solution. Any solution satisfies the integral equation

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s.$$

Suppose there are two solutions, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. One has that

$$\boldsymbol{x}_2(t) - \boldsymbol{x}_1(t) = \int_0^t (\boldsymbol{f}(\boldsymbol{x}_2(s)) - \boldsymbol{f}(\boldsymbol{x}_1(s)))\,\mathrm{d}s.$$

Since $\boldsymbol{f}$ is locally Lipschitz, there is an $L > 0$ and $\epsilon > 0$ such that as long as $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t) \in B(\boldsymbol{x}_0, \epsilon)$, then $|\boldsymbol{f}(\boldsymbol{x}_2(s)) - \boldsymbol{f}(\boldsymbol{x}_1(s))| \le L|\boldsymbol{x}_2(s) - \boldsymbol{x}_1(s)|$. By Theorem 1.1 there is a $\delta > 0$ such that this condition holds for $t \in (-\delta, \delta)$. One then has that

$$|\boldsymbol{x}_2(t) - \boldsymbol{x}_1(t)| \le L \int_0^t |\boldsymbol{x}_2(s) - \boldsymbol{x}_1(s)|\,\mathrm{d}s.$$

By Lemma 1.11 this yields

$$|\boldsymbol{x}_2(t) - \boldsymbol{x}_1(t)| \le 0 \cdot \mathrm{e}^{Lt};$$

hence, $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t)$ for all t. □

1.3. Continuity with respect to initial data

The following result indicates that under the assumption leading to unique solutions, the solution set is continuous with respect to variations in the initial data.

Theorem 1.17 (Continuity with respect to initial conditions)**.** *Suppose that $\boldsymbol{f}$ is locally Lipschitz. Consider the two initial value problems*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}),\ \boldsymbol{x}(0) = \boldsymbol{a}; \quad \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}),\ \boldsymbol{x}(0) = \boldsymbol{a}+\boldsymbol{h}.$$

Denote the solutions by $\boldsymbol{x}_0(t)$ and $\boldsymbol{x}_{\boldsymbol{h}}(t)$, respectively. For each $\epsilon > 0$ there is a $\delta > 0$ and $L_\epsilon > 0$ such that for $|\boldsymbol{h}| < \epsilon$ one has

$$|\boldsymbol{x}_0(t) - \boldsymbol{x}_{\boldsymbol{h}}(t)| \le \epsilon \mathrm{e}^{L_\epsilon t}$$

for all $t \in (-\delta, \delta)$.

Proof: Similar to that for Theorem 1.16. □

Regarding unique solutions to equation (1.2), one has the following useful properties.

Lemma 1.18. *Denote the solution to equation (1.2) by $\phi(t; x_0)$. For any fixed $t_0 \in \mathbb{R}$,*

$$\phi(t; x_0) = \phi(t - t_0; \phi(t_0; x_0)).$$

Proof: Set $s = t - t_0$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}s};$$

hence, the form of equation (1.2) does not change, so that both $\phi(t; x_0)$ and $\phi(t - t_0; \phi(t_0; x_0))$ are solutions. Since $\phi(0; x_0) = \phi(-t_0; \phi(t_0; x_0))$, by uniqueness they must be on the same trajectory. □

Corollary 1.19. *Suppose that there is a $T > 0$ such that the solution satisfies $\phi(T) = x_0$ with $\phi(t) \neq x_0$ for all $0 < t < T$. Then $\phi(t + T) = \phi(t)$ for all $t > 0$, i.e., $\phi(t)$ is a periodic orbit.*

Proof: By Lemma 1.18 the solution satisfies

$$\phi(t; x_0) = \phi(t - T; \phi(T; x_0)) = \phi(t - T; x_0).$$ □

1.4. Extensibility

Suppose that in equation (1.2) that $\boldsymbol{f}$ is C^1 on the open set G. Denote the unique solution by $\phi(t)$, and suppose that $J := (\alpha, \beta)$ is the maximal interval of existence. We must now understand what happens to the solution as $t \to \alpha^+$ and $t \to \beta^-$.

Theorem 1.20 (Extensibility theorem)**.** *For each compact set $K \subset G$ there is a $t \in J$ such that $\phi(t) \notin K$; thus,*

$$\lim_{t\to\beta^-} \phi(t) \in \partial G, \quad \lim_{t\to\alpha^+} \phi(t) \in \partial G.$$

In particular, if $G = \mathbb{R}^n$, then

$$\lim_{t\to\beta^-} |\phi(t)| = \lim_{t\to\alpha^+} |\phi(t)| = +\infty.$$

Proof: Suppose that there is a compact set $K \subset G$ such that $\phi(t) \in K$ for all $t \in J$. Since $[0, \beta] \times K$ is compact, there is an $M > 0$ such that $|\boldsymbol{f}(\boldsymbol{x})| \leq M$ for all $(t, \boldsymbol{x}) \in [0, \beta] \times K$. Let $s_1, s_2 \in [0, \beta)$ be chosen so that $s_1 < s_2$. One has that

$$|\phi(s_2) - \phi(s_1)| \leq \int_{s_1}^{s_2} |\boldsymbol{f}(\phi(s))| \,\mathrm{d}s \leq M|s_2 - s_1|;$$

hence, ϕ is uniformly continuous on $[0, \beta)$. As a consequence, $\phi(t)$ extends continuously to $[0, \beta]$; in particular, $\phi(\beta) = \lim_{t\to\beta^-} \phi(t)$ exists, and

$$\phi(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{f}(\phi(s)) \,\mathrm{d}s, \quad t \in [0, \beta].$$

Now consider the IVP

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(\beta) = \phi(\beta).$$

Since $K \subset G$, there is a $\delta > 0$ such that a solution is defined on the interval $(\beta - \delta, \beta + \delta)$. Denote this solution by $\psi(t)$, and note that by uniqueness $\psi(t) = \phi(t)$ for $t \in (\beta - \delta, \beta]$. Set

$$\gamma(t) = \begin{cases} \phi(t), & t \in [0, \beta) \\ \psi(t), & t \in [\beta, \beta + \delta). \end{cases}$$

It is easy to check that $\gamma(t)$ solves the original IVP on the interval $[0, \beta + \delta)$. Hence, the interval J is not maximal. □

Finally reconsider the result of Theorem 1.20 in the case that $G = \mathbb{R}^n$. As a consequence of the next result, it can be assumed for theoretical purposes that in this situation the solution to equation (1.2) exists for all time.

Corollary 1.21. *Without loss of generality, if $\boldsymbol{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is continuous, then solutions to equation (1.2) exist for all $t \in \mathbb{R}$.*

Proof: Let $g : G \subset \mathbb{R}^n \mapsto \mathbb{R}^+$ be smooth. It will first be shown that the trajectories to $\dot{x} = \boldsymbol{f}(\boldsymbol{x})$ are the same as those to $\boldsymbol{x}' = g(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})$. Let $\phi(t)$ be a solution to $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. Set

$$r(t) := \int_0^t \frac{\mathrm{d}s}{g(\phi(s))},$$

so that $\dot{r} = 1/g(\phi(t)) > 0$. By the chain rule,

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{g}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}r},$$

so that

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \Longleftrightarrow \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}r} = g(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}).$$

Thus, $\phi(r)$ is a solution to $\boldsymbol{x}' = g(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})$.

In particular, set $g(\boldsymbol{x}) := 1/(1 + |\boldsymbol{f}(\boldsymbol{x})|^2) \le 1$. By Corollary 1.3 the solution to $\boldsymbol{x}' = g(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})$ is given by

$$\phi(t) = \boldsymbol{x}_0 + \int_0^t \frac{\boldsymbol{f}(\phi(s))}{1 + |\boldsymbol{f}(\phi(s))|^2}\,\mathrm{d}s,$$

which yields the estimate

$$|\phi(t)| \le |\boldsymbol{x}_0| + \int_0^t 1\,\mathrm{d}s = |\boldsymbol{x}_0| + |t|.$$

By Theorem 1.20 the solution then exists for all $t \in \mathbb{R}$. □

1.5. Examples

Herein a few examples are considered which illustrate the utility of the above theory.

Example. Suppose that $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz with Lipschitz constant L. Since $|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(0)| \le L|\boldsymbol{x}|$ implies that $|\boldsymbol{f}(\boldsymbol{x})| \le |\boldsymbol{f}(0)| + L|\boldsymbol{x}|$, a solution to equation (1.2) satisfies for $t > 0$,

$$|\boldsymbol{x}(t)| \le |\boldsymbol{x}_0| + \int_0^t |\boldsymbol{f}(\boldsymbol{x}(s))|\,\mathrm{d}s \le |\boldsymbol{x}_0| + |\boldsymbol{f}(0)|t + L\int_0^t |\boldsymbol{x}(s)|\,\mathrm{d}s.$$

Applying Gronwall's inequality implies that

$$|\boldsymbol{x}(t)| \le (|\boldsymbol{x}_0| + |\boldsymbol{f}(0)|t)\mathrm{e}^{Lt},$$

i.e., the solution is uniformly bounded on $[0, T]$ for any $T > 0$. Since $T > 0$ is arbitrary, the solution is defined on $[0, +\infty)$. Reversing time and applying the same argument yields that the solution is defined on $(-\infty, +\infty)$.

In particular, when $\boldsymbol{f}$ is linear in $\boldsymbol{x}$, i.e., $\boldsymbol{f}(t, \boldsymbol{x}) = \boldsymbol{A}(t)\boldsymbol{x}$, then the result of Theorem 1.20 can be improved via the above argument.

Corollary 1.22. *Consider*

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0,$$

where $\boldsymbol{A}(t) \in \mathbb{R}^{n\times n}$ is continuous on (a, b). If $t_0 \in (a, b)$, then there is a unique solution defined on (a, b).

Proof: Let $\epsilon > 0$ be given, and suppose that $t_0 \in (a+\epsilon, b-\epsilon)$. The solution $\phi(t)$ satisfies

$$\phi(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{A}(s)\phi(s)\,\mathrm{d}s.$$

Since $\boldsymbol{A}$ is continuous, there is an $L > 0$ such that $|\boldsymbol{A}(t)| \le L$ for all $t \in [a+\epsilon, b-\epsilon]$. Thus, after applying Gronwall's inequality one gets

$$|\phi(t)| \le |\boldsymbol{x}_0|\mathrm{e}^{L|t-t_0|},$$

i.e., the solution is uniformly bounded on $(a+\epsilon, b-\epsilon)$. By Theorem 1.20 the solution exists for all $t \in [a+\epsilon, b-\epsilon]$. Since $\epsilon > 0$ is arbitrary, the solution exists on (a, b). □

Example. Consider

$$\dot{x} = h(t)g(x), \quad x(t_0) = x_0,$$

where $t_0, x_0 > 0$ and g and h are positive C^1 functions defined for $t \ge 0$ and $x > 0$. Applying separation of variables, the unique solution $\phi(t)$ satisfies

$$\int_{x_0}^{\phi(t)} \frac{\mathrm{d}x}{g(x)} = \int_{t_0}^{t} h(s)\,\mathrm{d}s.$$

Since h is continuous for $t \ge 0$, one has that

$$\int_0^b h(s)\,\mathrm{d}s < +\infty$$

for any $b \ge 0$. Now assume that

$$\lim_{\epsilon\to 0^+} \int_{\epsilon}^{1} \frac{\mathrm{d}x}{g(x)} = +\infty. \tag{1.9}$$

If $\lim_{t\to a^+} \phi(t) = 0$ for some $a \ge 0$, then

$$\lim_{t\to a^+} \int_{x_0}^{\phi(t)} \frac{\mathrm{d}x}{g(x)} = \int_{t_0}^{a} h(s)\,\mathrm{d}s,$$

which is a contradiction. Hence, under the assumption given in equation (1.9) one has that $\lim_{t\to a^+} \phi(t) > 0$ for any $a \ge 0$, so that the solution exists for $t \in [0, t_0]$.

2. Linear systems

In this section we will concerned with solving linear systems of the type

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}, \tag{2.1}$$

where either $\boldsymbol{A}(t) \equiv \boldsymbol{A}$, $\boldsymbol{A}(t+T) = \boldsymbol{A}(t)$, or $\lim_{t\to+\infty} \boldsymbol{A}(t) = \boldsymbol{A}$. These are the cases which arise most frequently in applications. In general, a thorough understanding of the solution behavior to equation (2.1) is necessary before attempting to understand

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{f}(t,\boldsymbol{x}) \tag{2.2}$$

(see Lemma 2.4).

2.1. General results

Recall from Corollary 1.22 that if $\boldsymbol{A}(t)$ is is continuous on $I := (a,b)$, and if $t_0 \in I$, then for each $\boldsymbol{x}_0 \in \mathbb{R}^n$ a unique solution to equation (2.1) exists for all $t \in I$. A set of n linearly independent solutions, if it exists, is called a fundamental set of solutions. Let $\boldsymbol{e}_i \in \mathbb{R}^n$ for $i = 1,\dots,n$ denote the usual basis vectors. Pick $t_0 \in I$. By Corollary 1.22 one knows that for each i there exists a solution $\phi_i(t)$ defined on I such that $\phi_i(t_0) = \boldsymbol{e}_i$. If there exist scalars $a_1,\dots,a_n$ such that

$$a_1\phi_1(t) + \cdots + a_n\phi_n(t) \equiv \boldsymbol{0},$$

then in particular one must have that $\sum_i a_i\boldsymbol{e}_i = \boldsymbol{0}$. This is a contradiction; hence, the solutions are linearly independent. If one sets

$$\Phi(t) := (\phi_1(t)\ \dots\ \phi_n(t)) \in \mathbb{R}^{n\times n},$$

and notes that

$$\dot{\Phi} = (\dot{\phi}_1\ \dots\ \dot{\phi}_n) = (\boldsymbol{A}\phi_1\ \dots\ \boldsymbol{A}\phi_n) = \boldsymbol{A}\Phi,$$

one has that Φ is a matrix-valued solution to equation (2.1); furthermore, it satisfies the initial condition $\Phi(t_0) = \mathbb{1}$. Such a matrix-valued solution to equation (2.1) is called the principal fundamental matrix solution.

As a consequence of the above discussion we have the following representation for solutions to equation (2.1). The simple proof is left for the interested student.

Lemma 2.1. *The solution to equation (2.1) is given by* $\boldsymbol{x}(t) = \Phi(t)\boldsymbol{x}_0$.

The next result gives the first useful property of fundamental matrix solutions (e.g., see [**17**, Theorem 6.6]).

Lemma 2.2 (Liouville's (Abel's) formula)**.** *If $\Phi(t)$ is a fundamental matrix solution to equation (2.1), then*

$$\det(\Phi(t)) = \det(\Phi(t_0))\mathrm{e}^{\int_{t_0}^t \mathrm{trace}(\boldsymbol{A}(s))\,\mathrm{d}s}.$$

As a consequence of Abel's formula, one has that if the fundamental matrix solution is nonsingular at one point in time, then it is nonsingular as long as it is defined.

Corollary 2.3. $\Phi(t)$ *is a fundamental matrix solution if and only if* $\Phi(t_0)$ *is nonsingular.*

Example. Let $\Phi(t)$ be the principle fundamental matrix solution to equation (2.1) at $t = t_0$. Suppose that

$$\lim_{t\to+\infty}\int_{t_0}^t \mathrm{trace}(\boldsymbol{A}(s))\,\mathrm{d}s \ge -M > -\infty.$$

It will now be shown that there is at least one solution to equation (2.1) which is nonzero in the limit $t \to +\infty$. As a consequence of Liouville's formula,

$$\lim_{t\to+\infty}\det(\Phi(t)) = \lim_{t\to+\infty}\mathrm{e}^{\int_{t_0}^t \mathrm{trace}(\boldsymbol{A}(s))\,\mathrm{d}s} \ge \mathrm{e}^{-M}.$$

Suppose that for $\phi_i(t) := \Phi(t)\boldsymbol{e}_i$ one has that $\lim_{t\to+\infty} |\phi_i(t)| = 0$ for all i. This necessarily implies that $\lim_{t\to+\infty} \det(\Phi(t)) = 0$, which is a contradiction. Hence, there exists a j such that $\lim_{t\to+\infty} |\phi_j(t)| \neq 0$. However, the converse is not true. Consider

$$\boldsymbol{A} := \mathrm{diag}(-2, 1).$$

Then $\int_0^t \mathrm{trace}(\boldsymbol{A}(s))\,\mathrm{d}s = -t \to -\infty$; however, both solutions to equation (2.1) in this case do not approach zero.

Now consider

$$\Psi(t) := \Phi(t)\boldsymbol{B}, \tag{2.3}$$

where $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ is constant and nonsingular. By the product rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi\boldsymbol{B}) = \dot{\Phi}\boldsymbol{B} = \boldsymbol{A}(t)(\Phi\boldsymbol{B});$$

hence, $\Psi(t)$ is a matrix-valued solution to equation (2.1) which satisfies the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{B}$. Such a solution is called a fundamental matrix solution. Note that as a consequence of the existence and uniqueness theorems all nonsingular matrix-valued solutions to equation (2.1) are of the form given in equation (2.3).

Finally, and as expected, the solution to equation (2.1) is crucial in constructing solutions to equation (2.2). The result of Corollary 1.3 yields one solution formula; however, it ignores the effect that the linear solution has on the nonlinear perturbation. The below result gives a formulation which is more convenient in applications, and which will be used throughout this text.

Lemma 2.4 (Variation of constants formula)**.** *Consider*

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{f}(t, \boldsymbol{x}), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0.$$

If $\Phi(t)$ represents the principal fundamental matrix solution to equation (2.1), then the solution is given by

$$\boldsymbol{x}(t) = \Phi(t)\boldsymbol{x}_0 + \Phi(t)\int_{t_0}^t \Phi(s)^{-1}\boldsymbol{f}(s, \boldsymbol{x}(s))\,\mathrm{d}s.$$

Proof: It is clear that $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. Differentiating yields

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \dot{\Phi}(t)\boldsymbol{x}_0 + \dot{\Phi}(t)\int_{t_0}^t \Phi(s)^{-1}\boldsymbol{f}(s, \boldsymbol{x}(s))\,\mathrm{d}s + \Phi(t)(\Phi(t)^{-1}\boldsymbol{f}(t, \boldsymbol{x}(t)) \\ &= \boldsymbol{A}(t)\Phi(t)(\boldsymbol{x}_0 + \int_{t_0}^t \Phi(s)^{-1}\boldsymbol{f}(s, \boldsymbol{x}(s))\,\mathrm{d}s) + \boldsymbol{f}(t, \boldsymbol{x}(t)) \\ &= \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{f}(t, \boldsymbol{x}(t)).\end{aligned}$$

□

Remark 2.5. As a consequence of Lemma 2.4 and Gronwall's inequality one may expect that a detailed analysis of the solution behavior for equation (2.1) will yield definitive information regarding the solutions to equation (2.2).

2.2. Equations with constant coefficients

In the previous subsection general results concerning solutions to equation (2.1) were given. However, knowing that a solution exists is not generally sufficient when attempting to understand the solution behavior to equation (2.2). In order to answer concrete questions associated with equation (2.2), it is necessary to understand the solution behavior to equation (2.1) in great detail. This necessitates that one restrict the form that $\boldsymbol{A}$ is allowed to take. In this subsection it will be assumed that $\boldsymbol{A}$ is actually a constant matrix.

2.2.1. The fundamental matrix solution

Consider the partial sum

$$\boldsymbol{S}_N(t) := \sum_{n=0}^{N} \boldsymbol{A}^n \frac{t^n}{n!}, \quad N \in \mathbb{N}_0,$$

and note that $\boldsymbol{S}_N(0) = \mathbb{1}$ for each $N \in \mathbb{N}_0$. For each N one has that $\boldsymbol{S}_N$ is smooth in t; furthermore, since $|\boldsymbol{A}\boldsymbol{B}| \le |\boldsymbol{A}|\,|\boldsymbol{B}|$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$, one has that for each $n \in \mathbb{N}$,

$$\left| \boldsymbol{A}^n \frac{t^n}{n!} \right| \le |\boldsymbol{A}|^n \frac{T^n}{n!}, \quad |t| \le T,$$

which yields the estimate

$$\begin{aligned} |\boldsymbol{S}_N(t)| &\le \sum_{n=0}^{N} |\boldsymbol{A}|^n \frac{|t|^n}{n!} \\ &\le \sum_{n=0}^{\infty} |\boldsymbol{A}|^n \frac{|t|^n}{n!} \\ &\le \mathrm{e}^{|\boldsymbol{A}|T}, \quad |t| \le T. \end{aligned} \tag{2.4}$$

In order to continue, the following version of the Weierstrass M-test is required.

Lemma 2.6 (Weierstrass M-test)**.** *Let $\boldsymbol{S}_N : \mathbb{R} \mapsto \mathbb{R}^{n\times n}$ for $N \in \mathbb{N}_0$ be such that $|\boldsymbol{S}_N(t)| \le M_T$ for $|t| \le T$. The sequence $\{\boldsymbol{S}_N(t)\}$ converges absolutely and uniformly for $|t| \le T$.*

As a consequence of the Weierstrass M-test and the estimate in equation (2.4) one has that for each fixed $T \in \mathbb{R}^+$ the partial sums $\boldsymbol{S}_N(t)$ converge absolutely and uniformly for $|t| \le T$. Since T is arbitrary, the sums converge for all $t \in \mathbb{R}$. Furthermore, since each $\boldsymbol{S}_N(t)$ is smooth, one has that the limit is continuous in t.

Definition 2.7. For $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ set

$$\mathrm{e}^{\boldsymbol{A}t} := \lim_{N\to+\infty} \boldsymbol{S}_N(t) = \sum_{n=0}^{\infty} \boldsymbol{A}^n \frac{t^n}{n!}.$$

It will now be shown that in the case of constant matrices, $\mathrm{e}^{\boldsymbol{A}t}$ is a fundamental matrix solution to equation (2.1). One has that for each $N \in \mathbb{N}_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{S}_N(t) = \boldsymbol{A}\boldsymbol{S}_{N-1}(t) = \boldsymbol{S}_{N-1}(t)\boldsymbol{A}.$$

The right-hand side follows from the product rule and the fact that $\boldsymbol{A}\boldsymbol{A}^k = \boldsymbol{A}^k\boldsymbol{A}$ for each $k \in \mathbb{N}_0$. By taking the limit of $N \to +\infty$ and using the fact that the convergence is uniform one has that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\boldsymbol{A}t} = \boldsymbol{A}\mathrm{e}^{\boldsymbol{A}t} = \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{A}.$$

Upon noting that $\mathrm{e}^{\boldsymbol{A}\cdot 0} = \mathbb{1}$ one has the following result.

Lemma 2.8. *The principal fundamental matrix solution to $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ at $t = 0$ is $\mathrm{e}^{\boldsymbol{A}t}$.*

It will now be shown that $\mathrm{e}^{\boldsymbol{A}t}$ satisfies the usual properties associated with the exponential function. First, let $s \in \mathbb{R}$ be given, and consider the initial value problem

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x}(0) = \mathrm{e}^{\boldsymbol{A}s}.$$

As a consequence of Lemma 2.8 and the discussion in Section 2.1 one knows that the unique solution is given by $\mathrm{e}^{\boldsymbol{A}t}\mathrm{e}^{\boldsymbol{A}s}$. Now set $\Psi(t) := \mathrm{e}^{\boldsymbol{A}(s+t)}$, and note by the chain rule and Lemma 2.8 that

$$\dot{\Psi} = \boldsymbol{A}\Psi, \quad \Psi(0) = \mathrm{e}^{\boldsymbol{A}s}.$$

By uniqueness one can then conclude that

$$e^{\boldsymbol{A}(s+t)} = e^{\boldsymbol{A}t}e^{\boldsymbol{A}s}.$$

Upon switching s and t one then gets that

$$e^{\boldsymbol{A}(s+t)} = e^{\boldsymbol{A}(t+s)} = e^{\boldsymbol{A}s}e^{\boldsymbol{A}t};$$

hence,

$$e^{\boldsymbol{A}(s+t)} = e^{\boldsymbol{A}t}e^{\boldsymbol{A}s} = e^{\boldsymbol{A}s}e^{\boldsymbol{A}t}. \tag{2.5}$$

Note that upon setting $s = -t$ in equation (2.5) one gets that

$$\left(e^{\boldsymbol{A}t}\right)^{-1} = e^{-\boldsymbol{A}t}.$$

Now suppose that $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ is such that $\boldsymbol{AB} = \boldsymbol{BA}$. One then has that $\boldsymbol{B}\boldsymbol{S}_N(t) = \boldsymbol{S}_N(t)\boldsymbol{B}$; hence, upon taking the limit one has $\boldsymbol{B}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{B}$. By the product rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{\boldsymbol{A}t}e^{\boldsymbol{B}t}) = \boldsymbol{A}e^{\boldsymbol{A}t}e^{\boldsymbol{B}t} + e^{\boldsymbol{A}t}\boldsymbol{B}e^{\boldsymbol{B}t} = (\boldsymbol{A}+\boldsymbol{B})e^{\boldsymbol{A}t}e^{\boldsymbol{B}t}.$$

Note that the above argument also yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{\boldsymbol{B}t}e^{\boldsymbol{A}t}) = (\boldsymbol{A}+\boldsymbol{B})e^{\boldsymbol{A}t}e^{\boldsymbol{B}t}.$$

However, by Lemma 2.8 one also has that

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{(\boldsymbol{A}+\boldsymbol{B})t} = (\boldsymbol{A}+\boldsymbol{B})e^{\boldsymbol{A}+\boldsymbol{B}t};$$

hence, the by uniqueness of solutions one has that

$$e^{(\boldsymbol{A}+\boldsymbol{B})t} = e^{\boldsymbol{A}t}e^{\boldsymbol{B}t} = e^{\boldsymbol{B}t}e^{\boldsymbol{A}t}.$$

Finally, suppose that $\boldsymbol{B}$ is nonsingular. Since $\boldsymbol{B}^{-1}\boldsymbol{A}^n\boldsymbol{B} = (\boldsymbol{B}^{-1}\boldsymbol{AB})^n$ for each $n \in \mathbb{N}_0$, one has that

$$\boldsymbol{B}^{-1}\boldsymbol{S}_N(t)\boldsymbol{B} = \sum_{n=0}^{N}\frac{t^n}{n!}(\boldsymbol{B}^{-1}\boldsymbol{AB})^n.$$

Taking the limit of $N \to +\infty$ and using Definition 2.7 gives

$$\boldsymbol{B}^{-1}e^{\boldsymbol{A}t}\boldsymbol{B} = e^{\boldsymbol{B}^{-1}\boldsymbol{AB}t}.$$

The above argument can be summarized as following:

Lemma 2.9. *Suppose that $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$, and that $s, t \in \mathbb{R}$.. Then*

(a) $(e^{\boldsymbol{A}t})^{-1} = e^{-\boldsymbol{A}t}$

(b) $e^{\boldsymbol{A}(s+t)} = e^{\boldsymbol{A}s}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}e^{\boldsymbol{A}s}$

(c) *if $\boldsymbol{AB} = \boldsymbol{BA}$, then* $e^{(\boldsymbol{A}+\boldsymbol{B})t} = e^{\boldsymbol{A}t}e^{\boldsymbol{B}t} = e^{\boldsymbol{B}t}e^{\boldsymbol{A}t}$

(d) *if $\boldsymbol{B}$ is nonsingular, then* $\boldsymbol{B}e^{\boldsymbol{A}t}\boldsymbol{B}^{-1} = e^{\boldsymbol{BAB}^{-1}t}$.

We will now consider some special cases in which $e^{\boldsymbol{A}t}$ can be easily computed.

Lemma 2.10. *Suppose that $\boldsymbol{A} = \boldsymbol{P}\Lambda\boldsymbol{P}^{-1}$, where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and $\boldsymbol{P} = (\boldsymbol{v}_1 \cdots \boldsymbol{v}_n)$ with $\boldsymbol{A}\boldsymbol{v}_k = \lambda_k\boldsymbol{v}_k$. Then $e^{\boldsymbol{A}t} = \boldsymbol{P}\,\mathrm{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})\boldsymbol{P}^{-1}$.*

Proof: By Lemma 2.9(d) one has that

$$\mathrm{e}^{\boldsymbol{A}t}=\boldsymbol{P}\mathrm{e}^{\Lambda t}\boldsymbol{P}^{-1};$$

hence, it is enough to compute $\mathrm{e}^{\Lambda t}$. Since $\Lambda^k=\mathrm{diag}(\lambda_1^k,\dots,\lambda_n^k)$, one gets that

$$\mathrm{e}^{\Lambda t}=\mathrm{diag}(\sum_{k=0}^{\infty}\frac{(\lambda_1 t)^k}{k!},\cdots\sum_{k=0}^{\infty}\frac{(\lambda_n t)^k}{k!})=\mathrm{diag}(\mathrm{e}^{\lambda_1 t},\dots,\mathrm{e}^{\lambda_n t}).\qquad\square$$

Lemma 2.11. *If $\boldsymbol{A}=\lambda\mathbb{1}+\boldsymbol{N}$, then $\mathrm{e}^{\boldsymbol{A}t}=\mathrm{e}^{\lambda t}\mathrm{e}^{\boldsymbol{N}t}$.*

Proof: Since $\mathbb{1}\boldsymbol{N}=\boldsymbol{N}\mathbb{1}$, one can immediately apply Lemma 2.9(c). Implicit in the calculation is the identity $\mathrm{e}^{\lambda t\mathbb{1}}=\mathrm{e}^{\lambda t}\mathbb{1}$. $\square$

Finally, suppose that

$$\boldsymbol{A}=a\mathbb{1}-b\boldsymbol{J},\quad \boldsymbol{J}:=\begin{pmatrix}0&1\\-1&0\end{pmatrix},$$

where $a,b\in\mathbb{R}$. The eigenvalues of $\boldsymbol{A}$ are $a\pm\mathrm{i}b$; furthermore, by Lemma 2.11 one has that $\mathrm{e}^{\boldsymbol{A}t}=\mathrm{e}^{at}\mathrm{e}^{-b\boldsymbol{J}t}$. It can be checked that

$$\begin{aligned}\mathrm{e}^{-b\boldsymbol{J}t}&=\begin{pmatrix}\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(bt)^{2n}}{(2n)!} & \displaystyle-\sum_{n=0}^{\infty}(-1)^n\frac{(bt)^{2n+1}}{(2n+1)!}\\ \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(bt)^{2n+1}}{(2n+1)!} & \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(bt)^{2n}}{(2n)!}\end{pmatrix}\\&=\begin{pmatrix}\cos(bt)&-\sin(bt)\\ \sin(bt)&\cos(bt)\end{pmatrix};\end{aligned}$$

hence,

$$\mathrm{e}^{\boldsymbol{A}t}=\mathrm{e}^{at}\begin{pmatrix}\cos(bt)&-\sin(bt)\\ \sin(bt)&\cos(bt)\end{pmatrix}.$$

As a generalization, suppose that $\boldsymbol{A}\in\mathbb{R}^{2n\times2n}$ has complex eigenvalues $\lambda_j=a_j+\mathrm{i}b_j$ for $j=1,\dots,n$. If the eigenvectors $\boldsymbol{w}_j=\boldsymbol{u}_j+\mathrm{i}\boldsymbol{v}_j$ are such that $\{\boldsymbol{u}_1,\dots,\boldsymbol{u}_n,\boldsymbol{v}_1,\dots,\boldsymbol{v}_n\}$ is a linearly independent set, then as a consequence of [**15**, Chapter 1.6] it is then known that for

$$\boldsymbol{P}:=(\boldsymbol{v}_1\,\boldsymbol{u}_1\,\dots\,\boldsymbol{v}_n\,\boldsymbol{u}_n)$$

one has

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}=\Lambda,$$

where

$$\Lambda:=\mathrm{diag}\left(\begin{pmatrix}a_1&-b_1\\b_1&a_1\end{pmatrix},\dots,\begin{pmatrix}a_n&-b_n\\b_n&a_n\end{pmatrix}\right).\tag{2.6}$$

As a consequence of the above discussion,

$$\mathrm{e}^{\Lambda t}=\mathrm{diag}\left(\mathrm{e}^{a_1t}\begin{pmatrix}\cos(b_1t)&-\sin(b_1t)\\ \sin(b_1t)&\cos(b_1t)\end{pmatrix},\dots,\mathrm{e}^{a_nt}\begin{pmatrix}\cos(b_nt)&-\sin(b_nt)\\ \sin(b_nt)&\cos(b_nt)\end{pmatrix}\right).\tag{2.7}$$

Upon using Lemma 2.9(d), the following lemma has now been proved.

Lemma 2.12. *Suppose that $\boldsymbol{A}=\boldsymbol{P}\Lambda\boldsymbol{P}^{-1}$, where Λ is given in equation (2.6). Then*

$$\mathrm{e}^{\boldsymbol{A}t}=\boldsymbol{P}\mathrm{e}^{\Lambda t}\boldsymbol{P}^{-1},$$

where $\mathrm{e}^{\Lambda t}$ is given in equation (2.7).

Example. Suppose that

$$\boldsymbol{A} := \begin{pmatrix} -4 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{pmatrix}.$$

The eigenvalues and associated eigenvectors are

$$\lambda_1 = -4,\ \boldsymbol{v}_1 = (1,0,0)^{\mathrm{T}}; \quad \lambda_2 = 2+\mathrm{i},\ \boldsymbol{w}_2 = (0,1+\mathrm{i},1)^{\mathrm{T}}.$$

Setting $\boldsymbol{P} = (\boldsymbol{v}_1\ \mathrm{Im}\,\boldsymbol{w}_2\ \mathrm{Re}\,\boldsymbol{w}_2)$ yields that

$$\Lambda = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Thus,

$$\mathrm{e}^{\Lambda t} = \begin{pmatrix} \mathrm{e}^{-4t} & 0 & 0 \\ 0 & \mathrm{e}^{2t}\cos t & -\mathrm{e}^{2t}\sin t \\ 0 & \mathrm{e}^{2t}\sin t & \mathrm{e}^{2t}\cos t \end{pmatrix},$$

and $\mathrm{e}^{\boldsymbol{A}t} = \boldsymbol{P}\mathrm{e}^{\Lambda t}\boldsymbol{P}^{-1}$.

2.2.2. The Jordan canonical form

The results of Lemma 2.10 and Lemma 2.12 required that one be able to diagonalize $\boldsymbol{A}$ in a particular manner. If $\boldsymbol{A}$ is symmetric, or if more generally the eigenvectors form a basis for $\mathbb{C}^n$, then $\boldsymbol{A}$ can be diagonalized in such a way. However, there are special cases, which are typically bifurcation points in parameter space, for which the diagonalization assumption leading to Lemma 2.10 and Lemma 2.12 breaks down. It is this case which will be covered in this subsection. Herein it will be assumed that all of the eigenvalues of $\boldsymbol{A}$ are real-valued. The case of complex eigenvalues is covered in [**15**, Section 1.8].

Definition 2.13. The spectrum of $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is given by

$$\sigma(\boldsymbol{A}) := \{\lambda \in \mathbb{C} : \det(\boldsymbol{A} - \lambda\mathbb{1}) = 0\}.$$

The multiplicity of $\lambda \in \sigma(\boldsymbol{A})$ is the order of the zero of $\det(\boldsymbol{A} - \lambda\mathbb{1}) = 0$.

Definition 2.14. Let $\lambda \in \sigma(\boldsymbol{A})$ have multiplicity p. Let $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ for $1 \le m \le p$ form a basis for $\ker(\boldsymbol{A} - \lambda\mathbb{1})$. Then

(a) m is the geometric multiplicity λ, i.e., $m_{\mathrm{g}}(\lambda) = m$

(b) p is the algebraic multiplicity of λ, i.e., $m_{\mathrm{a}}(\lambda) = m$.

$\lambda \in \sigma(\boldsymbol{A})$ is simple if $m_{\mathrm{g}}(\lambda) = m_{\mathrm{a}}(\lambda) = 1$, and $\lambda \in \sigma(\boldsymbol{A})$ is semi-simple if $m_{\mathrm{g}}(\lambda) = m_{\mathrm{a}}(\lambda) \ge 2$.

Remark 2.15. If $m_{\mathrm{g}}(\lambda) = m_{\mathrm{a}}(\lambda)$ for each $\lambda \in \sigma(\boldsymbol{A})$, i.e., if each eigenvalue is semi-simple, then $\boldsymbol{A}$ is diagonalizable.

Definition 2.16. Let $\lambda \in \sigma(\boldsymbol{A})$ be such that a.m.$= p$. For $k = 2, \dots, p$, any nonzero solution $\boldsymbol{v}$ of $(\boldsymbol{A} - \lambda\mathbb{1})^k\boldsymbol{v} = \boldsymbol{0}$ with $(\boldsymbol{A} - \lambda\mathbb{1})^{k-1}\boldsymbol{v} \ne \boldsymbol{0}$ is a generalized eigenvector.

Remark 2.17. Note that $\ker((\boldsymbol{A} - \lambda\mathbb{1})^j) \subset \ker((\boldsymbol{A} - \lambda\mathbb{1})^{j+1})$ for any $j \in \mathbb{N}$.

In order to better illustrate the above ideas, consider the following generic example. First suppose that

$$\boldsymbol{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \lambda\mathbb{1} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

One has that $\lambda \in \sigma(\boldsymbol{A})$ satisfies $m_{\mathrm{g}}(\lambda) = 1$ and $m_{\mathrm{a}}(\lambda) = 2$. The eigenvector is $\boldsymbol{v} = (1,0)^{\mathrm{T}}$, and the generalized eigenvector is $\boldsymbol{w} = (0,1)^{\mathrm{T}}$. To generalize, for a given $j \ge 2$ let $\boldsymbol{N} \in \mathbb{R}^{j\times j}$ satisfy

$$(\boldsymbol{N})_{\ell m} := \begin{cases} 1, & m = \ell + 1,\ \ell = 1, \dots, j-1 \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

It can be checked that $\boldsymbol{N}$ is a nilpotent matrix of order j, i.e., $\boldsymbol{N}^j = \boldsymbol{0}$ and $\boldsymbol{N}^{j-1} \neq \boldsymbol{0}$; hence,

$$\mathrm{e}^{\boldsymbol{N}t} = \sum_{n=0}^{j-1} \frac{t^n}{n!} \boldsymbol{N}^n.$$

Note that in this case the entries of $\mathrm{e}^{\boldsymbol{N}t}$ are polynomials of degree no larger than $j-1$. For the matrix $\boldsymbol{A} = \lambda\mathbb{1} + \boldsymbol{N}$ one has that $\lambda \in \sigma(\boldsymbol{A})$ with $m_{\mathrm{g}}(\lambda) = 1$ and $m_{\mathrm{a}}(\lambda) = j$. Finally, from Lemma 2.11 one has that

$$\mathrm{e}^{\boldsymbol{A}t} = \mathrm{e}^{\lambda t}\mathrm{e}^{\boldsymbol{N}t} = \mathrm{e}^{\lambda t} \sum_{n=0}^{j-1} \frac{t^n}{n!} \boldsymbol{N}_j^n.$$

In order to put a matrix in Jordan canonical form, the general idea is to first build chains of eigenvectors using the above ideas. This construction will not be carried out herein (e.g., see [**15**, Section 1.8] and the references therein), as the procedure is quite technical, and all that is necessary in the subsequent sections is the final result. As is seen below, the key to systematically diagonalize a nondiagonalizable matrix is to use nilpotent matrices of the form given in equation (2.8). As a result of the above discussion, note that for each Jordan block in the statement of Theorem 2.18 one has $m_{\mathrm{g}}(\lambda) = 1$.

Theorem 2.18 (Jordan canonical form)**.** *Suppose that $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ has real eigenvalues $\lambda_1, \dots, \lambda_n$. There exists a basis of generalized eigenvectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ such that with $\boldsymbol{P} := (\boldsymbol{v}_1 \cdots \boldsymbol{v}_n)$ one has that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \mathrm{diag}(\boldsymbol{B}_1, \dots, \boldsymbol{B}_r)$, where the Jordan blocks are of the form $\boldsymbol{B}_j = \lambda\mathbb{1} + \boldsymbol{N} \in \mathbb{R}^{\ell\times\ell}$ for some $1 \leq \ell \leq n$, and $\boldsymbol{N}$ is given in equation (2.8).*

As a consequence of Theorem 2.18, Lemma 2.10, Lemma 2.11, and Lemma 2.12 one has the following result.

Theorem 2.19. *Every entry of $\mathrm{e}^{\boldsymbol{A}t}$ is composed of linear combinations of $p(t)\mathrm{e}^{\alpha t}\cos\beta t$ and $p(t)\mathrm{e}^{\alpha t}\sin\beta t$, where $\lambda = \alpha + \mathrm{i}\beta \in \sigma(\boldsymbol{A})$ and $p(t)$ is a polynomial of degree no larger than $n-1$.*

Example. Consider

$$\boldsymbol{A} = \begin{pmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here $m_{\mathrm{g}}(-1) = 1$ and $m_{\mathrm{a}}(-1) = 2$, while $m_{\mathrm{g}}(1) = m_{\mathrm{a}}(1) = 1$. It can be checked that with

$$\boldsymbol{v}_1 = (0,2,1)^{\mathrm{T}}, \quad \boldsymbol{v}_2 = (1,0,0)^{\mathrm{T}}, \quad \boldsymbol{v}_3 = (0,1,0)^{\mathrm{T}},$$

upon setting $\boldsymbol{P} = (\boldsymbol{v}_1\,\boldsymbol{v}_2\,\boldsymbol{v}_3)$ one has

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \mathrm{diag}(\boldsymbol{B}_1, \boldsymbol{B}_2); \qquad \boldsymbol{B}_1 = (1), \quad \boldsymbol{B}_2 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

One then has that

$$\mathrm{e}^{\boldsymbol{A}t} = \boldsymbol{P} \begin{pmatrix} \mathrm{e}^{t} & 0 & 0 \\ 0 & \mathrm{e}^{-t} & t\mathrm{e}^{-t} \\ 0 & 0 & \mathrm{e}^{-t} \end{pmatrix} \boldsymbol{P}^{-1}.$$

2.2.3. Estimates on solutions

Now that the construction of the principal fundamental matrix solution is understood, it is time to see how one can estimate the solution behavior. The following preliminary result is first needed.

Proposition 2.20. *For each $\epsilon > 0$ and each $j \in \mathbb{N}$ there exists an $M(j,\epsilon) > 0$ such that*

$$t^j \leq M(j,\epsilon)\mathrm{e}^{\epsilon t}, \quad M(j,\epsilon) = \left(\frac{j}{e\epsilon}\right)^j.$$

for all $t \geq 0$.

Proof: Set $g(t) := t^j e^{-\epsilon t}$. Since $g(t)$ is continuous and satisfies $g(t) \to 0$ as $t \to +\infty$, there exists a $M(j,\epsilon) > 0$ such that $g(t) \le M(j,\epsilon)$ for all $t \ge 0$. The upper bound is found by finding the maximum of $g(t)$. $\square$

Now, by Theorem 2.19 each entry of $e^{\boldsymbol{A}t}$ is composed of linear combinations of terms like $p(t)e^{\alpha t}\cos\beta t$ and $p(t)e^{\alpha t}\sin\beta t$, where $\alpha + i\beta \in \sigma(\boldsymbol{A})$ and $p(t)$ is a polynomial of degree no greater than $n-1$. The next result then immediately follows from Proposition 2.20.

Lemma 2.21. *Set* $\sigma_{\mathrm{M}} := \max\{\mathrm{Re}\,\lambda : \lambda \in \sigma(\boldsymbol{A})\}$. *For each* $\epsilon > 0$ *there exists an* $M(\epsilon) > 0$ *such that*

$$|e^{\boldsymbol{A}t}| \le M(\epsilon)e^{(\sigma_{\mathrm{M}}+\epsilon)t}$$

for all $t \ge 0$.

If all $\lambda \in \sigma(\boldsymbol{A})$ are semi-simple, then in Theorem 2.19 one can set $p(t) \equiv 1$. In Proposition 2.20, if one sets $j = 0$, then one has $\epsilon = 0$. These observations yield the following refinement of Lemma 2.21.

Corollary 2.22. *If all* $\lambda \in \sigma(\boldsymbol{A})$ *are semi-simple, then one can set* $\epsilon = 0$ *in Lemma 2.21.*

Lemma 2.21 sets an upper bound on the growth rate of $|e^{\boldsymbol{A}t}|$. The below results sets a lower bound. Note that it does not depend on the multiplicity of the eigenvalue with minimal real part.

Lemma 2.23. *If* $\sigma_{\mathrm{m}} := \min\{\mathrm{Re}\,\lambda : \lambda \in \sigma(\boldsymbol{A})\}$, *then*

$$|e^{\boldsymbol{A}t}| \ge e^{\sigma_{\mathrm{m}}t}$$

for all $t \ge 0$.

Proof: Let $\lambda \in \sigma(\boldsymbol{A})$ be such that $\mathrm{Re}\,\lambda = \sigma_{\mathrm{m}}$, and let $\boldsymbol{v}$ be the associated eigenvector. One has that $e^{\boldsymbol{A}t}\boldsymbol{v} = e^{\lambda t}\boldsymbol{v}$, so that

$$\frac{|e^{\boldsymbol{A}t}\boldsymbol{v}|}{|\boldsymbol{v}|} = e^{\sigma_{\mathrm{m}}t}.$$

The result now follows from the definition of the matrix norm. $\square$

Let us now refine the above estimates. In particular, we wish to find invariant subspaces which have proscribed behavior for solutions residing in them. Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, and let $\lambda \in \sigma(\boldsymbol{A})$ be such that $m_{\mathrm{g}}(\lambda) = m$ and $m_{\mathrm{a}}(\lambda) = p \ge m$. As a consequence of the reduction of $\boldsymbol{A}$ to Jordan canonical form one has there exists a basis of generalized eigenvectors $\{\boldsymbol{v}_{j,i}\}$ such that with $\boldsymbol{v}_{0,i} := \boldsymbol{0}$,

$$(\boldsymbol{A} - \lambda\mathbb{1})\boldsymbol{v}_{j,i} = \boldsymbol{v}_{j-1,i}, \quad i = 1,\dots,m,\ j = 1,\dots,a_i, \quad \sum_{i=1}^{m} a_i = p.$$

Definition 2.24. For each $\lambda \in \sigma(\boldsymbol{A})$ set $E_\lambda := \mathrm{span}(\{\boldsymbol{v}_{j,i}\})$. The subspace E_λ is the called the generalized eigenspace of $\boldsymbol{A}$ corresponding to the eigenvalue λ.

Proposition 2.25. E_λ *is invariant under multiplication by* $\boldsymbol{A}$, *i.e.,* $\boldsymbol{A}E_\lambda \subset E_\lambda$.

Proof: Let $\boldsymbol{v} \in E_\lambda$ be given, so that $v = \sum_{i,j} c_{j,i}\boldsymbol{v}_{j,i}$ for some constants $c_{i,j}$. By linearity one has that

$$\boldsymbol{A}\boldsymbol{v} = \sum_{i,j} c_{j,i}\boldsymbol{A}\boldsymbol{v}_{j,i} = \sum_{i,j} c_{j,i}(\lambda\boldsymbol{v}_{j,i} + \boldsymbol{v}_{j-1,i}) = \lambda\boldsymbol{v} + \sum_{i,j} c_{j,i}\boldsymbol{v}_{j-1,i}.$$

Thus, $\boldsymbol{A}\boldsymbol{v} \in E_\lambda$. $\square$

As a consequence of the definition of $e^{\boldsymbol{A}t}$ and Proposition 2.25 one has the following result.

Corollary 2.26. $e^{\boldsymbol{A}t}E_\lambda \subset E_\lambda$.

Proof: By Proposition 2.25 $\boldsymbol{A}E_\lambda \subset E_\lambda$, which by an induction argument yields that $\boldsymbol{A}^k E_\lambda \subset E_\lambda$ for each $k \ge 1$. The result now follows from the series representation for $e^{\boldsymbol{A}t}$ and using the fact that E_λ is a closed subspace. $\square$

The following spectral subsets given below will be used to decompose $\mathbb{R}^n$:

Definition 2.27. The stable spectrum and subspace are given by

$$\sigma^{\mathrm{s}}(\boldsymbol{A}) := \{\lambda \in \sigma(\boldsymbol{A}) \,:\, \operatorname{Re}\lambda < 0\}, \quad E^{\mathrm{s}} := \{\oplus E_{\lambda_j} \,:\, \lambda_j \in \sigma^{\mathrm{s}}(\boldsymbol{A})\},$$

the unstable spectrum and subspace are given by

$$\sigma^{\mathrm{u}}(\boldsymbol{A}) := \{\lambda \in \sigma(\boldsymbol{A}) \,:\, \operatorname{Re}\lambda > 0\}, \quad E^{\mathrm{u}} := \{\oplus E_{\lambda_j} \,:\, \lambda_j \in \sigma^{\mathrm{u}}(\boldsymbol{A})\},$$

and the center spectrum and subspace are given by

$$\sigma^{\mathrm{c}}(\boldsymbol{A}) := \{\lambda \in \sigma(\boldsymbol{A}) \,:\, \operatorname{Re}\lambda = 0\}, \quad E^{\mathrm{c}} := \{\oplus E_{\lambda_j} \,:\, \lambda_j \in \sigma^{\mathrm{c}}(\boldsymbol{A})\}.$$

Proposition 2.28. *One has that:*

(a) $\mathbb{R}^n = E^{\mathrm{s}} \oplus E^{\mathrm{u}} \oplus E^{\mathrm{c}}$

(b) $\dim(E^{\mathrm{s,u,c}})$ *is the number of generalized eigenvectors in the basis for* $E^{\mathrm{s,u,c}}$*, respectively*

(c) $\mathrm{e}^{\boldsymbol{A}t}E^{\mathrm{s,u,c}} \subset E^{\mathrm{s,u,c}}$*, respectively.*

Proof: By construction E^{s}, E^{u}, and E^{c} are mutually disjoint. The statement of (a) then follows from the fact that the generalized eigenvectors form a basis of $\mathbb{R}^n$. Since $\boldsymbol{A}E_{\lambda_j} \subset E_{\lambda_j}$ for each $\lambda_j \in \sigma^{\mathrm{s,u,c}}(\boldsymbol{A})$, parts (b) and (c) follow from Corollary 2.26 and linearity. □

Armed with Proposition 2.28, one can use Lemma 2.21 and Lemma 2.23 to describe the behavior associated with solutions residing in each invariant subspace $E^{\mathrm{s,u,c}}$.

Theorem 2.29. *If* $\boldsymbol{x}_0 \in E^{\mathrm{s}}$*, then there exist positive constants* $c < a$ *and* $m, M \geq 1$ *such that*

$$m\mathrm{e}^{-at}|\boldsymbol{x}_0| \leq |\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}_0| \leq M\mathrm{e}^{-ct}|\boldsymbol{x}_0|,$$

while if $\boldsymbol{x}_0 \in E^{\mathrm{u}}$*, then there exist positive constants* $c < a$ *and* $m, M \geq 1$ *such that*

$$m\mathrm{e}^{ct}|\boldsymbol{x}_0| \leq |\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}_0| \leq M\mathrm{e}^{at}|\boldsymbol{x}_0|.$$

Finally, if $\boldsymbol{x}_0 \in E^{\mathrm{c}}$*, then there exists a* $k \in \mathbb{N}_0$ *with* $0 \leq k \leq n-1$ *such that*

$$m|\boldsymbol{x}_0| \leq |\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}_0| \leq M(1+|t|^k)|\boldsymbol{x}_0|.$$

Remark 2.30. If all $\lambda \in \sigma^{\mathrm{c}}(\boldsymbol{A})$ are semi-simple, then $k = 0$; hence, in this case solutions residing in E^{c} are uniformly bounded.

Proof: The result will be proven only for $\boldsymbol{x}_0 \in E^{\mathrm{s}}$, as the other proofs are similar. Set

$$\sigma^- := \min\{\operatorname{Re}\lambda \,:\, \lambda \in \sigma^{\mathrm{s}}(\boldsymbol{A})\}, \quad \sigma^+ := \max\{\operatorname{Re}\lambda \,:\, \lambda \in \sigma^{\mathrm{s}}(\boldsymbol{A})\},$$

and note that $\sigma^- \leq \sigma^+ < 0$. By Proposition 2.28 $\mathrm{e}^{\boldsymbol{A}t}E^{\mathrm{s}} \subset E^{\mathrm{s}}$; furthermore, by Lemma 2.21 and Lemma 2.23 for each $0 < \epsilon < |\sigma_+|$ there is an $M(\epsilon)$ such that

$$m\mathrm{e}^{\sigma_- t}|\boldsymbol{x}_0| \leq |\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}_0| \leq M(\epsilon)\mathrm{e}^{(\sigma_+ + \epsilon)t}|\boldsymbol{x}_0|.$$ □

One of the implications of Theorem 2.29 is that the behavior for solutions residing in $E^{\mathrm{s,u}}$ is exponential in nature. Solutions in the unstable subspace exhibit growth for $t \geq 0$, while those solutions in the stable subspace decay for $t \geq 0$. The behavior of solutions in the center subspace is unknown without more detailed information, and all that can be said is that any temporal growth is polynomial in nature.

One can summarize in the following manner. By Proposition 2.28 one can write the initial data as

$$\boldsymbol{x}_0 = \boldsymbol{x}_0^{\mathrm{s}} + \boldsymbol{x}_0^{\mathrm{c}} + \boldsymbol{x}_0^{\mathrm{u}}, \quad \boldsymbol{x}_0^{\mathrm{s,c,u}} \in E^{\mathrm{s,c,u}}.$$

Using linearity then yields

$$\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}_0 = \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}_0^{\mathrm{s}} + \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}_0^{\mathrm{c}} + \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}_0^{\mathrm{u}}. \tag{2.9}$$

The solution behavior associated with $\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}_0^{\mathrm{s,c,u}}$ is given in Theorem 2.29. The result of equation (2.9), along with the definitions given in Definition 2.27 associated with the various subspaces, can be summarized in the following definition.

Definition 2.31. Consider
$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}.$$
The critical point $\boldsymbol{x} = \boldsymbol{0}$ is a

- sink (attractor): $\sigma(\boldsymbol{A}) = \sigma^{\mathrm{s}}(\boldsymbol{A})$
- source (repeller, negative attractor): $\sigma(\boldsymbol{A}) = \sigma^{\mathrm{u}}(\boldsymbol{A})$
- saddle (unstable saddle): $\sigma(\boldsymbol{A}) = \sigma^{\mathrm{s}}(\boldsymbol{A}) \cup \sigma^{\mathrm{u}}(\boldsymbol{A})$ with $\sigma^{\mathrm{s,u}}(\boldsymbol{A}) \neq \varnothing$.

If $\sigma^{\mathrm{c}}(\boldsymbol{A}) = \varnothing$, then the system is hyperbolic, and the associated flow $\phi_t(\boldsymbol{x}_0) := \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}_0$ is called a hyperbolic flow.

2.2.4. Linear perturbations: stable

Consider equation (2.2) under the assumption that $\boldsymbol{f}$ is smooth enough to ensure unique solutions. Recall the result of Lemma 2.4. Assuming that $\boldsymbol{A}(t) \equiv \boldsymbol{A}$, by using the results of Lemma 2.8 and Lemma 2.9 one can reformulate the solution as
$$\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_0 + \int_{t_0}^{t} \mathrm{e}^{\boldsymbol{A}(t-s)}\boldsymbol{f}(s, \boldsymbol{x}(s))\,\mathrm{d}s. \tag{2.10}$$
Before continuing, we need the following definition which characterizes the behavior of solutions near critical points, i.e., zeros of the vector field, for equation (2.2).

Definition 2.32. Consider equation (2.2). A critical point $\boldsymbol{a}$ is stable if for each $\epsilon > 0$ there is $\delta > 0$ such that if $\boldsymbol{x}_0 \in B(\boldsymbol{a}, \delta)$, then $\boldsymbol{x}(t) \in B(\boldsymbol{a}, \epsilon)$ for all $t \geq 0$. The critical point is asymptotically stable if it is stable and if $\lim_{t\to+\infty} \boldsymbol{x}(t) = \boldsymbol{a}$. If the critical point is not stable, it is unstable.

Remark 2.33. Consider the linear system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$. Upon applying the results of Theorem 2.29 it is not difficult to show that $\boldsymbol{x} = \boldsymbol{0}$ is

- unstable if $\sigma^{\mathrm{u}}(\boldsymbol{A}) \neq \varnothing$
- stable if $\sigma(\boldsymbol{A}) = \sigma^{\mathrm{s}}(\boldsymbol{A}) \cup \sigma^{\mathrm{c}}(\boldsymbol{A})$ and all $\lambda \in \sigma^{\mathrm{c}}(\boldsymbol{A})$ are semi-simple
- asymptotically stable if $\sigma(\boldsymbol{A}) = \sigma^{\mathrm{s}}(\boldsymbol{A})$.

Consider equation (2.2) under the assumptions that $\boldsymbol{A}(t) \equiv \boldsymbol{A}$ and
$$\boldsymbol{f}(t, \boldsymbol{x}) = \boldsymbol{B}(t)\boldsymbol{x}, \quad \int_0^\infty |\boldsymbol{B}(t)|\,\mathrm{d}t < \infty$$
First suppose that $\sigma(\boldsymbol{A}) = \sigma^{\mathrm{s}}(\boldsymbol{A}) \cup \sigma^{\mathrm{c}}(\boldsymbol{A})$, and that each $\lambda \in \sigma^{\mathrm{c}}(\boldsymbol{A})$ is semi-simple. It will be shown that the solution, which is given in equation (2.10), is bounded. By the assumption on $\sigma(\boldsymbol{A})$ one can apply Theorem 2.29 and conclude that there exists an $M > 0$ such that $|\mathrm{e}^{\boldsymbol{A}t}| \leq M$; hence, the solution satisfies the estimate
$$|\boldsymbol{x}(t)| \leq M|\boldsymbol{x}_0| + \int_{t_0}^{t} M|\boldsymbol{B}(s)|\,|\boldsymbol{x}(s)|\,\mathrm{d}s.$$
Upon using Gronwall's inequality one gets that
$$|\boldsymbol{x}(t)| \leq M|\boldsymbol{x}_0|\mathrm{e}^{M\int_{t_0}^{t}|\boldsymbol{B}(s)|\,\mathrm{d}s},$$
so by assumption there is a $C > 1$ such that $|\boldsymbol{x}(t)| \leq C|\boldsymbol{x}_0|$ for any $t \geq t_0$. In particular, this estimate shows that $\boldsymbol{x} = \boldsymbol{0}$ is stable.

Now suppose that $\sigma(\boldsymbol{A}) = \sigma^{\mathrm{s}}(\boldsymbol{A})$, so by Theorem 2.29 that there exists an $M, \alpha > 0$ such that $|\mathrm{e}^{\boldsymbol{A}t}| \leq M\mathrm{e}^{-\alpha t}$. As above, one then has that
$$|\boldsymbol{x}(t)| \leq M\mathrm{e}^{-\alpha(t-t_0)}|\boldsymbol{x}_0| + \int_{t_0}^{t} M\mathrm{e}^{-\alpha(t-s)}|\boldsymbol{B}(s)|\,|\boldsymbol{x}(s)|\,\mathrm{d}s.$$

Set $\boldsymbol{y}(t) := |\boldsymbol{x}(t)|\mathrm{e}^{\alpha t}$, so that the above can be rewritten as

$$|\boldsymbol{y}(t)| \le M|\boldsymbol{y}(t_0)| + \int_{t_0}^{t} M|\boldsymbol{B}(s)|\,|\boldsymbol{y}(s)|\,\mathrm{d}s.$$

As above, there is a $C > 1$ such that $|\boldsymbol{y}(t)| \le C|\boldsymbol{y}(t_0)|$, which implies that

$$|\boldsymbol{x}(t)| \le C|\boldsymbol{x}_0|\mathrm{e}^{-\alpha(t-t_0)}.$$

Hence, $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable.

2.2.5. Linear perturbations: unstable

Finally, suppose that $\sigma(\boldsymbol{A}) = \sigma^{\mathrm{s}}(\boldsymbol{A}) \cup \sigma^{\mathrm{u}}(\boldsymbol{A})$. In this case it will be shown that under the assumption that

$$\int_{-\infty}^{+\infty} |\boldsymbol{B}(t)|\,\mathrm{d}t < \infty,$$

as $t \to \pm\infty$ the system will have the same behavior as the unperturbed system. The following discussion has as its inspiration the work of [**5**, Lecture 4]. Before continuing, the following preparatory theorem is needed:

Theorem 2.34 (Banach's fixed point theorem)**.** *Let X be a complete normed vector space, and let $D \subset X$ be closed. Let $\mathcal{T} : D \mapsto D$ be such that for all $u, v \in D$,*

$$\|\mathcal{T}(u) - \mathcal{T}(v)\| \le L\|u - v\|, \quad 0 < L < 1;$$

in other words, assume that $\mathcal{T}$ is a contraction mapping. There is then a unique $u^* \in D$ such that $\mathcal{T}(u^*) = u^*$.

Proof: Let $u_0 \in D$ be given, and for $n \in \mathbb{N}$ define the sequence $\{u_n\}$ via $u_{n+1} = \mathcal{T}(u_n)$. First note that via an induction argument one gets

$$\|u_{n+1} - u_n\| = \|\mathcal{T}(u_n) - \mathcal{T}(u_{n-1})\| \le L\|u_n - u_{n-1}\| \le L^n\|u_1 - u_0\|.$$

This then implies that for each $k \in \mathbb{N}$,

$$\|u_{n+k} - u_n\| \le \sum_{j=0}^{k-1} \|u_{n+j-1} - u_{n+j}\| \le \sum_{j=0}^{k-1} L^{n+j}\|u_1 - u_0\| = L^n\|u_1 - u_0\| \sum_{j=0}^{k-1} L^j \le \frac{L^n}{1-L}\|u_1 - u_0\|.$$

Thus, $\{u_n\}$ is a Cauchy sequence, and since X is complete and D is closed one has a $u^* \in D$ such that $u_n \to u^*$. Since $\mathcal{T}$ is continuous, one then has that $\mathcal{T}(u^*) = u^*$. Lastly, u^* is unique, for if there exists another fixed point v^*, then

$$\|u^* - v^*\| = \|\mathcal{T}(u^*) - \mathcal{T}(v^*)\| \le L\|u^* - v^*\|,$$

which is a contradiction, as $L < 1$. $\square$

Remark 2.35. An application of Theorem 2.34 allows for a much easier proof of Theorem 1.16 for the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{2.11}$$

Let $\epsilon, L \in \mathbb{R}^+$ be such that for all $\boldsymbol{u}, \boldsymbol{v} \in B(\boldsymbol{x}_0, 2\epsilon)$, $|\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{v})| \le L|\boldsymbol{u} - \boldsymbol{v}|$. Now set

$$C = \sup\{|\boldsymbol{f}(\boldsymbol{x})| : \boldsymbol{x} \in B(\boldsymbol{x}_0, 2\epsilon)\}, \quad \delta := \min\{\frac{\epsilon}{C}, \frac{1}{2L}\},$$

and define

$$X := C^0([-\delta, \delta]; \mathbb{R}^n), \quad \|\boldsymbol{x}\| := \sup_{|t|\le\delta} |\boldsymbol{x}(t)|,$$

and

$$D := \{\boldsymbol{x} \in X \,:\, \sup_{|t|\le\delta} |\boldsymbol{x}(t) - \boldsymbol{x}_0| \le 2\epsilon\}.$$

If one now considers the mapping $\mathcal{T} \,:\, D \mapsto X$ given by

$$\mathcal{T}(\boldsymbol{x}) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{f}(\boldsymbol{x}(t))\,\mathrm{d}t, \quad |t| \le \delta,$$

then it can be shown that $\mathcal{T}$ satisfies the conditions given in Theorem 2.34; hence, there exists a unique solution to equation (2.11) The remaining details will be left for the interested student.

Let $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ be such that

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \Lambda := \mathrm{diag}(\boldsymbol{A}^{\mathrm{s}}, \boldsymbol{A}^{\mathrm{u}}),$$

where $\boldsymbol{A}^{\mathrm{s,u}} \in \mathbb{R}^{(n_\mathrm{s},n_\mathrm{u})\times(n_\mathrm{s},n_\mathrm{u})}$ with $n_\mathrm{s} + n_\mathrm{u} = n$, and $\sigma(\boldsymbol{A}^{\mathrm{s,u}}) = \sigma^{\mathrm{s,u}}(\boldsymbol{A}^{\mathrm{s,u}})$. Setting $\boldsymbol{x} = \boldsymbol{P}\boldsymbol{y}$ transforms the original system to

$$\dot{\boldsymbol{y}} = \Lambda\boldsymbol{y} + \widetilde{\boldsymbol{B}}(t)\boldsymbol{y}, \quad \widetilde{\boldsymbol{B}}(t) := \boldsymbol{P}^{-1}\boldsymbol{B}(t)\boldsymbol{P}. \tag{2.12}$$

Define the projection operators $\Pi_{\mathrm{s,u}}$ by

$$\Pi_\mathrm{s} := \mathrm{diag}(\mathbb{1}_\mathrm{s}, \boldsymbol{0}), \quad \Pi_\mathrm{u} := \mathrm{diag}(\boldsymbol{0}, \mathbb{1}_\mathrm{u}),$$

where $\mathbb{1}_{\mathrm{s,u}} \in \mathbb{R}^{(n_\mathrm{s},n_\mathrm{u})\times(n_\mathrm{s},n_\mathrm{u})}$. Note that the projection operators satisfy the properties

$$\Pi_\mathrm{s}\Pi_\mathrm{u} = \Pi_\mathrm{u}\Pi_\mathrm{s} = \boldsymbol{0}, \quad \Pi_{\mathrm{s,u}}^2 = \Pi_{\mathrm{s,u}}, \quad \Pi_\mathrm{s} + \Pi_\mathrm{u} = \mathbb{1}; \tag{2.13}$$

furthermore,

$$\Pi_{\mathrm{s,u}}\mathrm{e}^{\Lambda t} = \mathrm{e}^{\Lambda t}\Pi_{\mathrm{s,u}}. \tag{2.14}$$

As a consequence of Theorem 2.29 one has that there exists a $C, \alpha, \beta \in \mathbb{R}^+$ such that

$$|\mathrm{e}^{\Lambda t}\Pi_\mathrm{s}| \le C\mathrm{e}^{-\alpha t},\ t \ge 0; \quad |\mathrm{e}^{\Lambda t}\Pi_\mathrm{u}| \le C\mathrm{e}^{\beta t},\ t \le 0. \tag{2.15}$$

Choose $t_0 \in \mathbb{R}^+$ sufficiently large so that

$$\theta := C\int_{t_0}^{+\infty} |\widetilde{\boldsymbol{B}}(t)\|\,\mathrm{d}t < 1,$$

and set

$$X := C^0([t_0, +\infty); \mathbb{R}^n), \quad \|\boldsymbol{y}\| := \sup_{t\ge t_0} |\boldsymbol{y}(t)|.$$

Define the mapping

$$\mathcal{T}\boldsymbol{y} := \int_{t_0}^t \mathrm{e}^{\Lambda(t-s)}\Pi_\mathrm{s}\widetilde{\boldsymbol{B}}(s)\boldsymbol{y}(s)\,\mathrm{d}s - \int_t^{+\infty} \mathrm{e}^{\Lambda(t-s)}\Pi_\mathrm{u}\widetilde{\boldsymbol{B}}(s)\boldsymbol{y}(s)\,\mathrm{d}s. \tag{2.16}$$

It is not difficult to show that $\mathcal{T} : X \mapsto X$ with $\|\mathcal{T}\boldsymbol{y}\| \le \theta\|\boldsymbol{y}\|$, and that for any $\boldsymbol{y}_1, \boldsymbol{y}_2 \in X$,

$$\begin{aligned}|\mathcal{T}\boldsymbol{y}_1 - \mathcal{T}\boldsymbol{y}_2| &\le C\int_{t_0}^{+\infty} |\widetilde{\boldsymbol{B}}(s)|\,|\boldsymbol{y}_1(s) - \boldsymbol{y}_2(s)|\,\mathrm{d}s \\ &\le \theta\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|.\end{aligned}$$

Consequently $\mathcal{T}$ is a contraction map on X.

Now let $\boldsymbol{y}_0 \in \mathbb{R}^n$ be given so that $\Pi_\mathrm{s}\boldsymbol{y}_0 = \boldsymbol{y}_0$, and consider the integral equation

$$\boldsymbol{y}(t) = \mathrm{e}^{\Lambda(t-t_0)}\boldsymbol{y}_0 + \mathcal{T}\boldsymbol{y}(t). \tag{2.17}$$

As a consequence of Theorem 2.34 there is a unique solution in X to equation (2.17); furthermore, it is easy to verify that any solution to equation (2.17) is also a solution to equation (2.12) with the initial condition

$$\boldsymbol{y}(t_0) = \boldsymbol{y}_0 - \int_{t_0}^{+\infty} \mathrm{e}^{\Lambda(t-s)}\Pi_\mathrm{u}\widetilde{\boldsymbol{B}}(s)\boldsymbol{y}(s)\,\mathrm{d}s.$$

Note that there is then a one-to-one mapping between bounded solutions of the unperturbed problem and those for the perturbed problem.

Now it must be shown that the solution $\boldsymbol{y}(t) \to \boldsymbol{0}$ exponentially fast as $t \to +\infty$. From equation (2.15) one has that for $t \geq t_0$,

$$\begin{aligned}|\mathcal{T}\boldsymbol{y}(t)|\mathrm{e}^{\alpha t} &\leq C\int_{t_0}^{t}|\widetilde{\boldsymbol{B}}(s)|\,|\boldsymbol{y}(s)|\mathrm{e}^{\alpha s}\,\mathrm{d}s + \int_{t}^{+\infty}\mathrm{e}^{(\alpha+\beta)(t-s)}|\widetilde{\boldsymbol{B}}(s)|\,|\boldsymbol{y}(s)|\mathrm{e}^{\alpha s}\,\mathrm{d}s \\ &\leq \theta\|\boldsymbol{y}(t)\mathrm{e}^{\alpha t}\|.\end{aligned}$$

Consequently, for $t \geq t_0$ the solution to equation (2.17) satisfies

$$\begin{aligned}|\boldsymbol{y}(t)|\mathrm{e}^{\alpha t} &\leq C\mathrm{e}^{\alpha t_0}|\boldsymbol{y}_0| + |\mathcal{T}\boldsymbol{y}(t)|\mathrm{e}^{\alpha t} \\ &\leq C\mathrm{e}^{\alpha t_0}|\boldsymbol{y}_0| + \theta\|\boldsymbol{y}(t)\mathrm{e}^{\alpha t}\|,\end{aligned}$$

i.e.,

$$\|\boldsymbol{y}(t)\mathrm{e}^{\alpha t}\| \leq C\mathrm{e}^{\alpha t_0}|\boldsymbol{y}_0| + \theta\|\boldsymbol{y}(t)\mathrm{e}^{\alpha t}\|.$$

This necessarily implies that

$$(1-\theta)\|\boldsymbol{y}(t)\mathrm{e}^{\alpha t}\| \leq C\mathrm{e}^{\alpha t_0}|\boldsymbol{y}_0|,$$

which in turn yields that

$$|\boldsymbol{y}(t)| \leq \frac{C}{1-\theta}|\boldsymbol{y}_0|\mathrm{e}^{-\alpha(t-t_0)}.$$

In conclusion, it has been demonstrated that there is a one-one mapping between exponentially decaying solutions for the perturbed problem and those for the unperturbed problem. Finally consider equation (2.12) under the time reversal $r := -t$. One then has

$$\boldsymbol{y}' = -\Lambda\boldsymbol{y} - \widetilde{\boldsymbol{B}}(-r)\boldsymbol{y}, \quad {}' := \frac{\mathrm{d}}{\mathrm{d}r}.$$

Since $\sigma(-\Lambda) = -\sigma(\Lambda)$, the above argument shows that there is a one-one mapping between exponentially decaying solutions as $r \to +\infty$, i.e., as $t \to -\infty$, for the perturbed problem and those for the unperturbed problem. Note here that $\boldsymbol{y}_0 \in E^{\mathrm{u}}$.

2.2.6. Nonlinear perturbations

In the previous example the behavior of solutions to equation (2.2) in the case of linear and asymptotically zero perturbations was considered. The next result deals with nonlinear perturbations which are small in a neighborhood of the origin.

Theorem 2.36. *Consider equation (2.2) under the assumptions that $\boldsymbol{A}(t) \equiv \boldsymbol{A}$ and $|\boldsymbol{f}(t,\boldsymbol{x})| = \mathcal{O}(|\boldsymbol{x}|^2)$ for $|\boldsymbol{x}| \leq \delta$. Suppose that $\sigma(\boldsymbol{A}) = \sigma^{\mathrm{s}}(\boldsymbol{A})$. There then exist constants C, μ, α such that if $|\boldsymbol{x}_0| \leq \mu$, then the solution satisfies $|\boldsymbol{x}(t)| \leq C\mathrm{e}^{-\alpha(t-t_0)}|\boldsymbol{x}_0|$ for all $t \geq t_0$. In particular, $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable.*

Proof: Since $\sigma(\boldsymbol{A}) = \sigma^{\mathrm{s}}(\boldsymbol{A})$, by Theorem 2.29 there exists a $C \geq 1$ and $\lambda > 0$ such that $|\mathrm{e}^{\boldsymbol{A}t}| \leq C\mathrm{e}^{-\lambda t}$. Since $|\boldsymbol{f}(t,\boldsymbol{x})| = \mathcal{O}(|\boldsymbol{x}|^2)$, there exists a $k > 0$ such that $|\boldsymbol{f}(t,\boldsymbol{x})| \leq k|\boldsymbol{x}|^2$. Fix $\epsilon > 0$ so that $\epsilon < \delta$ and $Ck\epsilon < \lambda$. Set $\alpha := \lambda - Ck\epsilon$ and $\mu := \epsilon/C$, and note that $\alpha > 0$ and $0 < \mu < \epsilon < \delta$.

If $|\boldsymbol{x}_0| < \mu$ there exists a $\tau > t_0$ such that the solution $\boldsymbol{x}(t)$ satisfies $|\boldsymbol{x}(t)| \leq \epsilon$ on the interval $I := \{t \in \mathbb{R} : t_0 \leq t \leq \tau\}$. This implies that for $t \in I$ one has that

$$|\boldsymbol{f}(t, x(t))| \leq k|\boldsymbol{x}(t)|^2 \leq k\epsilon|\boldsymbol{x}(t)|.$$

By equation (2.10) the solution is given by

$$\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_0 + \int_{t_0}^{t}\mathrm{e}^{\boldsymbol{A}(t-s)}\boldsymbol{f}(s,\boldsymbol{x}(s))\,\mathrm{d}s,$$

which yields that as long as $t \in I$,

$$|\boldsymbol{x}(t)| \leq C\mathrm{e}^{-\lambda(t-t_0)}|\boldsymbol{x}_0| + Ck\epsilon\int_{t_0}^{t}\mathrm{e}^{-\lambda(t-s)}|\boldsymbol{x}(s)|\,\mathrm{d}s.$$

Rearranging the above inequality gives

$$\mathrm{e}^{\lambda(t-t_0)}|\boldsymbol{x}(t)| \leq C|\boldsymbol{x}_0| + Ck\epsilon\int_{t_0}^{t}\mathrm{e}^{\lambda(s-t_0)}|\boldsymbol{x}(s)|\,\mathrm{d}s,$$

which by Gronwall's inequality yields

$$\mathrm{e}^{\lambda(t-t_0)}|\boldsymbol{x}(t)| \leq C|\boldsymbol{x}_0|\mathrm{e}^{Ck\epsilon(t-t_0)},$$

or finally,

$$|\boldsymbol{x}(t)| \leq C|\boldsymbol{x}_0|\mathrm{e}^{-\alpha(t-t_0)}.$$

If $|\boldsymbol{x}_0| < \mu$, then the above estimate yields that $|\boldsymbol{x}(t)| \leq \epsilon$ for all $t \in I$. By the extensibility Theorem 1.20 one then has that $\tau = +\infty$; furthermore, $\boldsymbol{x} = \boldsymbol{0}$ is stable. Since $\alpha > 0$, $x = 0$ is asymptotically stable. ☐

Now consider

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{2.18}$$

where $\boldsymbol{f} \in C^k(\mathbb{R}^n)$ for some $k \geq 2$. Suppose that $\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{0}$. Upon setting $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{a}$ equation (2.18) becomes $\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y})$, where $\boldsymbol{g}(\boldsymbol{y}) := \boldsymbol{f}(\boldsymbol{a}+\boldsymbol{y})$. Note that $\boldsymbol{g}(\boldsymbol{0}) = \boldsymbol{0}$; hence, without loss of generality one can always assume in equation (2.18) that $\boldsymbol{a} = \boldsymbol{0}$. Now, by Taylor's theorem one can write

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{r}(\boldsymbol{x}),$$

where $A := \mathrm{D}\boldsymbol{f}(\boldsymbol{0})$ and

$$\boldsymbol{r}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{A}\boldsymbol{x} = \left(\int_0^1 [\mathrm{D}\boldsymbol{f}(s\boldsymbol{x}) - \boldsymbol{A}]\,\mathrm{d}s\right)\boldsymbol{x}.$$

Since $\mathrm{D}\boldsymbol{f}(\cdot)$ is smooth, by the Mean Value Theorem

$$|\mathrm{D}\boldsymbol{f}(s\boldsymbol{x}) - \boldsymbol{A}| \leq |s\boldsymbol{x}|\sup_{\tau\in[0,1]}|\mathrm{D}^2\boldsymbol{f}(\tau\boldsymbol{x})| \leq |\boldsymbol{x}|\sup_{\tau\in[0,1]}|\mathrm{D}^2\boldsymbol{f}(\tau\boldsymbol{x})|;$$

furthermore, since $\mathrm{D}^2\boldsymbol{f}(\cdot)$ is continuous, there exists a $\delta > 0$ and a constant $k > 0$ such that

$$\sup_{\tau\in[0,1]}|\mathrm{D}^2\boldsymbol{f}(\tau\boldsymbol{x})| \leq k, \quad \boldsymbol{x} \in B(\boldsymbol{0},\delta).$$

Thus, $|\boldsymbol{r}(\boldsymbol{x})| = \mathcal{O}(|\boldsymbol{x}|^2)$ for $\boldsymbol{x} \in B(\boldsymbol{0},\delta)$. Applying Theorem 2.36 yields the following result.

Corollary 2.37. *Consider equation (2.18) where $\boldsymbol{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is smooth with $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$. Set $\boldsymbol{A} := \mathrm{D}\boldsymbol{f}(\boldsymbol{0})$. If $\sigma(\boldsymbol{A}) = \sigma^{\mathrm{s}}(\boldsymbol{A})$, then there exists an $\alpha > 0$ and a neighborhood U of $\boldsymbol{x} = \boldsymbol{0}$ such that if $\boldsymbol{x}_0 \in U$, then $|\boldsymbol{x}(t)| \leq C|\boldsymbol{x}_0|\mathrm{e}^{-\alpha t}$ for all $t \geq 0$*

Now consider equation (2.18) under the time reversal $s := -t$. One than has

$$\boldsymbol{x}' = -\boldsymbol{A}\boldsymbol{x} - \boldsymbol{r}(\boldsymbol{x}), \quad ' := \frac{\mathrm{d}}{\mathrm{d}s}.$$

Since $\sigma(-\boldsymbol{A}) = -\sigma(\boldsymbol{A})$, Corollary 2.37 now applies for $s \geq 0$, i.e., $t \leq 0$.

Corollary 2.38. *Suppose that $\sigma(\boldsymbol{A}) = \sigma^{\mathrm{u}}(\boldsymbol{A})$. The result of Corollary 2.37 is true for $t \leq 0$.*

2.3. Equations with periodic coefficients

Herein we will consider equation (2.1) under the condition that $\boldsymbol{A}(t)$ is continuous and T-periodic, i.e., $\boldsymbol{A}(t+T)=\boldsymbol{A}(t)$ for some $T\in\mathbb{R}^+$. In order to understand the issues involved, first consider

$$\dot{x}=a(t)x,\quad a(t+T)=a(t).$$

The fundamental matrix solution is given by $\Phi(t)=\mathrm{e}^{\int_0^t a(s)\,\mathrm{d}s}$. Setting

$$\alpha:=\frac{1}{T}\int_0^T a(s)\,\mathrm{d}s,\quad p(t):=\int_0^t (a(s)-\alpha)\,\mathrm{d}s$$

yields $\Phi(t)=P(t)\mathrm{e}^{\alpha t}$, where $P(t):=\mathrm{e}^{p(t)}$. Now,

$$\begin{aligned}p(t+T)&=\int_0^t (a(s)-\alpha)\,\mathrm{d}s+\int_t^{t+T}(a(s)-\alpha)\,\mathrm{d}s\\&=p(t)+\int_0^T(a(s)-\alpha)\,\mathrm{d}s\\&=p(t)\end{aligned}$$

so that the fundamental matrix solution is the product of a periodic function with an exponential function. The initial goal is to show that this property is true for systems. We need a preparatory lemma.

Lemma 2.39. *If $\boldsymbol{C}\in\mathbb{R}^{n\times n}$ is nonsingular, then there exists a $\boldsymbol{B}\in\mathbb{C}^{n\times n}$ such that $\mathrm{e}^{\boldsymbol{B}}=\boldsymbol{C}$.*

Proof: Let $\boldsymbol{J}$ be the Jordan canonical form of $\boldsymbol{C}$, i.e., $\boldsymbol{P}^{-1}\boldsymbol{C}\boldsymbol{P}=\boldsymbol{J}$. If $\mathrm{e}^{\boldsymbol{K}}=\boldsymbol{J}$, then $\mathrm{e}^{\boldsymbol{P}\boldsymbol{K}\boldsymbol{P}^{-1}}=\boldsymbol{C}$; hence, it can be assumed that $\boldsymbol{C}$ is in canonical form. Let $\lambda_1,\dots,\lambda_k\in\sigma(\boldsymbol{C})$ have multiplicities $n_1,\dots,n_k$. One has that $\boldsymbol{C}=\mathrm{diag}(\boldsymbol{C}_1,\dots,\boldsymbol{C}_k)$, where each $\boldsymbol{C}_j\in\mathbb{C}^{n_j\times n_j}$ with $\boldsymbol{C}_j=\lambda_j\mathbb{1}+\boldsymbol{N}$, where $\boldsymbol{N}$ is nilpotent of order n_j. Since $\boldsymbol{C}$ is nonsingular, $\lambda_j\neq 0$ for all j. Motivated by the fact that

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n},\quad |x|<1,$$

set

$$\boldsymbol{B}_j:=\ln\lambda_j\mathbb{1}+\boldsymbol{S}_j,\quad \boldsymbol{S}_j:=\sum_{n=1}^{n_j-1}\frac{(-1)^{n+1}}{n}\left(\frac{\boldsymbol{N}}{\lambda_j}\right)^n.$$

The sum is finite because $\boldsymbol{N}$ is nilpotent. As a consequence of Lemma 2.9(c) one has

$$\mathrm{e}^{\boldsymbol{B}_j}=\lambda_j\mathrm{e}^{\boldsymbol{S}_j}.$$

It can be shown [8, p. 61-62] that

$$\mathrm{e}^{\boldsymbol{S}_j}=\mathbb{1}+\frac{1}{\lambda_j}\boldsymbol{N};$$

hence, $\mathrm{e}^{\boldsymbol{B}_j}=\boldsymbol{C}_j$. Upon setting $\boldsymbol{B}:=\mathrm{diag}(\boldsymbol{B}_1,\dots,\boldsymbol{B}_k)$, one has that $\mathrm{e}^{\boldsymbol{B}}=\boldsymbol{C}$. □

Remark 2.40. The matrix $\boldsymbol{B}$ given in Lemma 2.39 is not unique, as one has the identity

$$\mathrm{e}^{\boldsymbol{B}+2\ell\pi\mathrm{i}\mathbb{1}}=\mathrm{e}^{\boldsymbol{B}}\mathrm{e}^{2\ell\pi\mathrm{i}\mathbb{1}}=\boldsymbol{C}\mathbb{1},\quad \ell\in\mathbb{Z}.$$

As the following result shows, the decomposition of the fundamental matrix solution given in the beginning of this section for a scalar problem also holds for systems.

Theorem 2.41 (Floquet's theorem)**.** *Consider equation (2.1), where $\boldsymbol{A}(t)\in\mathbb{R}^{n\times n}$ is continuous with $\boldsymbol{A}(t+T)=\boldsymbol{A}(t)$. If $\Phi(t)$ is a fundamental matrix solution, then there exists a $\boldsymbol{B}\in\mathbb{C}^{n\times n}$ such that*

$$\Phi(t)=\boldsymbol{P}(t)\mathrm{e}^{\boldsymbol{B}t},\quad \boldsymbol{P}(t+T)=\boldsymbol{P}(t).$$

Proof: Since $\boldsymbol{A}(t)$ is T-periodic, if $\Phi(t)$ is a fundamental matrix solution, then so is $\Phi(t+T)$. By the uniqueness of solutions one then has that

$$\Phi(t+T) = \Phi(t)\Phi(T).$$

By Lemma 2.39 there is a $\boldsymbol{B} \in \mathbb{C}^{n\times n}$ such that $\Phi(T) = \mathrm{e}^{\boldsymbol{B}T}$, which yields $\Phi(t+T) = \Phi(t)\mathrm{e}^{\boldsymbol{B}T}$. Upon setting $\boldsymbol{P}(t) := \Phi(t)\mathrm{e}^{-\boldsymbol{B}t}$, one has that

$$\boldsymbol{P}(t+T) = \Phi(t)\mathrm{e}^{\boldsymbol{B}T}\mathrm{e}^{-\boldsymbol{B}(t+T)} = \Phi(t)\mathrm{e}^{-\boldsymbol{B}t} = \boldsymbol{P}(t). \qquad \square$$

Definition 2.42. The monodromy operator for the fundamental matrix solution given in Theorem 2.41 is given by $\mathrm{e}^{\boldsymbol{B}T}$. $\lambda \in \sigma(\mathrm{e}^{\boldsymbol{B}T})$ is known as a Floquet multiplier, and $\mu \in \sigma(\boldsymbol{B})$ is a characteristic exponent.

If μ is a characteristic exponent, then $\lambda = \mathrm{e}^{\mu T}$ is a Floquet multiplier. Since $\boldsymbol{B}$ is not unique, the characteristic exponents are not unique. However, the Floquet multipliers are unique, as these are given by $\sigma(\Phi(T))$. As is the case for equation (2.1) when $\boldsymbol{A}$ is constant, one has that $\sigma(\boldsymbol{B})$ plays a significant role in the behavior of solutions.

Lemma 2.43. *If $\mu \in \sigma(\boldsymbol{B})$, then there exists a (possibly complex) solution to equation (2.1) of the form $\mathrm{e}^{\mu t}\boldsymbol{p}(t)$, where $\boldsymbol{p}(t)$ is T-periodic.*

Proof: Let $\Phi(t)$ be the principal fundamental matrix solution at $t = 0$, so by Theorem 2.41 one has that $\Phi(t) = \boldsymbol{P}(t)\mathrm{e}^{\boldsymbol{B}t}$ with $\boldsymbol{P}(0) = \mathbb{1}$. Since $\mu \in \sigma(\boldsymbol{B})$, there exists a $\boldsymbol{v}$ such that $\boldsymbol{B}\boldsymbol{v} = \mu\boldsymbol{v}$. Thus, $\boldsymbol{x}(t) := \Phi(t)\boldsymbol{v} = \mathrm{e}^{\mu t}\boldsymbol{P}(t)\boldsymbol{v}$, which proves the result upon setting $\boldsymbol{p}(t) := \boldsymbol{P}(t)\boldsymbol{v}$. $\square$

Note that the solution given in Lemma 2.43 satisfies the identity $\boldsymbol{x}(t+T) = \lambda\boldsymbol{x}(t)$, where $\lambda := \mathrm{e}^{\mu T} \in \sigma(\mathrm{e}^{\boldsymbol{B}T})$ is a Floquet multiplier. An induction argument then yields that $\boldsymbol{x}(t+nT) = \lambda^n\boldsymbol{x}(t)$ for any $n \in \mathbb{N}$. Now consider the sequence $\{\boldsymbol{x}_k\}$, where $\boldsymbol{x}_k := \boldsymbol{x}(kT) = \lambda^k\boldsymbol{x}(0)$. If $\boldsymbol{x}_j = \boldsymbol{x}_0$ for some $j \in \mathbb{N}_0$, then one has a jT-periodic solution to equation (2.1). The argument leading to this assertion is similar to that leading to Corollary 1.19, and is left to the interested student. Since the solution is uniformly bounded on $[0,T)$, if one wishes to understand the dynamics associated with equation (2.1) it is sufficient to look at the behavior of this sequence. However, this is equivalent to looking at the sequence $\{\lambda^k\}$. If $|\lambda| > 1$, then $|\lambda^k| = |\lambda|^k \to \infty$ exponentially fast; hence, the solution $\boldsymbol{x} = \boldsymbol{0}$ to equation (2.1) is unstable. If $|\lambda| < 1$, then $1 > |\lambda|^k \to 0$ exponentially fast, so that the solution $\boldsymbol{x} = \boldsymbol{0}$ to equation (2.1) is stable. Now suppose that $|\lambda| = 1$, which implies that $\lambda = \mathrm{e}^{\mathrm{i}2\pi\theta}$ for some $\theta \in [0,1)$. If θ is rational, i.e., $\theta = p/q$ for $p,q \in \mathbb{N}$ relatively prime, then there is the periodic sequence given by

$$\{1, \mathrm{e}^{\mathrm{i}2\pi\theta}, \mathrm{e}^{\mathrm{i}4\pi\theta}, \dots, \mathrm{e}^{\mathrm{i}2(q-1)\pi\theta}\},$$

so that $\boldsymbol{x}_q = \boldsymbol{x}_0$. Thus, there exists a qT-periodic solution to equation (2.1). If θ is irrational, then the orbit is dense on the circle $|\lambda| = 1$, and the orbit is uniformly bounded, but not periodic.

Lemma 2.44. *Let $\lambda_1, \dots, \lambda_n \in \sigma(\mathrm{e}^{\boldsymbol{B}T})$. Then*

(a) *$|\lambda_j| < 1$ for all j implies that $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable*

(b) *$|\lambda_j| > 1$ for some j implies that $\boldsymbol{x} = \boldsymbol{0}$ is unstable*

(c) *$|\lambda_j| \le 1$ for all j with $|\lambda_j| = 1$ being semi-simple implies that $\boldsymbol{x} = \boldsymbol{0}$ is stable*

Proof: Set $\boldsymbol{x} = \boldsymbol{P}(t)\boldsymbol{y}$, which gives

$$\dot{\boldsymbol{x}} = \dot{\boldsymbol{P}}\boldsymbol{y} + \boldsymbol{P}\dot{\boldsymbol{y}},$$

i.e.,

$$\dot{\boldsymbol{y}} = \boldsymbol{P}(t)^{-1}(\boldsymbol{A}(t)\boldsymbol{P}(t) - \dot{\boldsymbol{P}}(t))\boldsymbol{y}.$$

Since $\boldsymbol{P}(t) = \Phi(t)\mathrm{e}^{-\boldsymbol{B}t}$, one has that

$$\dot{\boldsymbol{P}}(t) = \boldsymbol{A}(t)\boldsymbol{P}(t) - \boldsymbol{P}(t)\boldsymbol{B},$$

so that upon substitution, $\dot{\boldsymbol{y}} = \boldsymbol{B}\boldsymbol{y}$. As a consequence of Theorem 2.29 the behavior of $\boldsymbol{y}(t)$ is determined by $\sigma(\boldsymbol{B})$. Now, if $\mu \in \sigma^{\mathrm{s,u,c}}(\boldsymbol{B})$, then $\lambda = \mathrm{e}^{\mu T} \in \sigma(\mathrm{e}^{\boldsymbol{B}T})$ satisfies $|\lambda| < 1$, $|\lambda| > 1$, $|\lambda| = 1$, respectively. Since $\boldsymbol{P}(t)$ is continuous and T-periodic, the result follows. $\square$

As a consequence of Lemma 2.44 one knows that the dynamical behavior is determined by $\sigma(\Phi(T))$. However, one cannot usually directly compute the Floquet multipliers, as the explicit form of the fundamental matrix is generally unknown. As the following result shows, one can still get some information regarding the multipliers.

Lemma 2.45. *If $\lambda_j = \mathrm{e}^{\mu_j T}$ are the Floquet multipliers, then*

(a) $\displaystyle\prod_{j=1}^{n} \lambda_j = \mathrm{e}^{\int_0^T \mathrm{trace}\, \boldsymbol{A}(s)\,\mathrm{d}s}$

(b) $\displaystyle\sum_{j=1}^{n} \mu_j = \frac{1}{T}\int_0^T \mathrm{trace}\, \boldsymbol{A}(s)\,\mathrm{d}s \pmod{\frac{2\pi\mathrm{i}}{T}}$

Proof: Upon using Abel's formula in Lemma 2.2, and assuming that $\boldsymbol{P}(0) = \mathbb{1} = \boldsymbol{P}(T)$,

$$\det \Phi(T) = \det \mathrm{e}^{\boldsymbol{B}T} = \mathrm{e}^{\int_0^T \mathrm{trace}\, \boldsymbol{A}(s)\,\mathrm{d}s}.$$

Part (a) follows from the fact that

$$\det \mathrm{e}^{\boldsymbol{B}T} = \prod_{j=1}^{n} \lambda_j,$$

while part (b) follows from

$$\prod_{j=1}^{n} \lambda_j = \mathrm{e}^{T\sum_{j=1}^{n}\mu_j}. \qquad \square$$

2.3.1. Example: periodic forcing

For the first example which illustrates the utility of Floquet theory, consider

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{b}(t), \tag{2.19}$$

where $\boldsymbol{A}(t) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b}(t)$ are T-periodic. One has the following result concerning the existence of T-periodic solutions to equation (2.19).

Lemma 2.46. *Let $\Phi(t)$ be the principal fundamental matrix solution for equation (2.19). If $1 \notin \sigma(\Phi(T))$, then there exists a unique T-periodic solution.*

Proof: By the variation of constants formula, the solution to equation (2.19) is given by

$$\boldsymbol{x}(t) = \Phi(t)\boldsymbol{x}_0 + \int_0^t \Phi(t)\Phi(s)^{-1}\boldsymbol{b}(s)\,\mathrm{d}s.$$

In order to have a periodic solution, one must have that $\boldsymbol{x}(T) = x_0$. Upon some algebraic manipulation this yields

$$(\mathbb{1} - \Phi(T))x_0 = \int_0^T \Phi(T)\Phi(s)^{-1}\boldsymbol{b}(s)\,\mathrm{d}s. \tag{2.20}$$

If $\lambda \in \sigma(\Phi(T))$, then $1-\lambda \in \sigma(\mathbb{1} - \Phi(T))$. Hence, if $1 \notin \sigma(\Phi(T))$, then $\mathbb{1} - \Phi(T)$ is nonsingular, and $\boldsymbol{x}_0$ is then uniquely given by

$$\boldsymbol{x}_0 = (\mathbb{1} - \Phi(T))^{-1}\int_0^T \Phi(T)\Phi(s)^{-1}\boldsymbol{b}(s)\,\mathrm{d}s. \qquad \square$$

In general, of course, it is difficult to compute the Floquet multipliers. However, if $\boldsymbol{A}(t) \equiv \boldsymbol{A}$, then one has that if $\lambda \in \sigma(\boldsymbol{A})$, then $\mathrm{e}^{\lambda T} \in \sigma(\mathrm{e}^{\boldsymbol{A}T})$. The following result then follows immediately from Lemma 2.46.

Corollary 2.47. *Consider equation (2.19) under the condition that $\boldsymbol{A}(t) \equiv \boldsymbol{A}$. If there exists no $\lambda \in \sigma(\boldsymbol{A})$ such that*

$$\lambda = \mathrm{i}\frac{2\pi\ell}{T}, \quad \ell \in \mathbb{Z}, \tag{2.21}$$

then there exists a unique T-periodic solution.

Remark 2.48. The condition in equation (2.21) is automatically satisfied if $\sigma^{\mathrm{c}}(\boldsymbol{A}) = \varnothing$.

Even if the condition of $1 \notin \sigma(\Phi(T))$ in Lemma 2.46 is removed, it may still be possible to find periodic solutions to equation (2.19). Upon using the fact that $\Phi(t)$ is nonsingular, equation (2.20) can be rewritten as

$$(\Phi^{-1}(T) - \mathbb{1})\boldsymbol{x}_0 = \int_0^T \Phi(s)^{-1}\boldsymbol{b}(s)\,\mathrm{d}s. \tag{2.22}$$

If $1 \in \sigma(\Phi(T))$, then a solution to equation (2.22) exists if and only if

$$\int_0^T \Phi(s)^{-1}\boldsymbol{b}(s)\,\mathrm{d}s \in \ker\left((\Phi(T)^{-1} - \mathbb{1})^{\mathrm{T}}\right)^{\perp}. \tag{2.23}$$

Now suppose that $\boldsymbol{A}(t) \equiv \boldsymbol{A}$, and further suppose that $T = 2\pi$. Assume that $\mathrm{i} \in \sigma(\boldsymbol{A})$ is a simple eigenvalue, and that $\mathrm{i}\ell \notin \sigma(\boldsymbol{A})$ for any other $\ell \in \mathbb{N}\backslash\{1\}$. Using Theorem 2.18, write

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \Lambda,$$

where there exist block matrices $\boldsymbol{B}_1, \dots, \boldsymbol{B}_r$ such that

$$\Lambda = \mathrm{diag}\left(-\boldsymbol{J}, \boldsymbol{B}_1, \dots, \boldsymbol{B}_r\right), \quad \boldsymbol{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The change of variables $\boldsymbol{y} = \boldsymbol{P}^{-1}\boldsymbol{x}$ allows one to rewrite equation (2.19) as

$$\dot{\boldsymbol{y}} = \Lambda\boldsymbol{y} + \boldsymbol{c}(t), \quad \boldsymbol{c}(t) := \boldsymbol{P}^{-1}\boldsymbol{b}(t).$$

Since

$$(\mathrm{e}^{-2\pi\Lambda} - \mathbb{1})^{\mathrm{T}} = \mathrm{diag}\left(\boldsymbol{0}, \mathrm{e}^{-2\pi\boldsymbol{B}_1^{\mathrm{T}}} - \mathbb{1}, \dots, \mathrm{e}^{-2\pi\boldsymbol{B}_r^{\mathrm{T}}} - \mathbb{1}\right),$$

(note that $(\mathrm{e}^{\boldsymbol{B}})^{\mathrm{T}} = \mathrm{e}^{\boldsymbol{B}^{\mathrm{T}}}$) one has the right-hand side of equation (2.23) satisfies

$$\ker\left((\mathrm{e}^{-2\pi\Lambda} - \mathbb{1})^{\mathrm{T}}\right)^{\perp} = \mathrm{span}\{\boldsymbol{e}_3, \dots, \boldsymbol{e}_n\}.$$

Thus, there exists a 2π-periodic solution to equation (2.19) if and only if

$$\boldsymbol{e}_j \cdot \int_0^{2\pi} \mathrm{e}^{-\Lambda s}\boldsymbol{c}(s)\,\mathrm{d}s = 0, \quad j = 1, 2. \tag{2.24}$$

Alternatively, since

$$\boldsymbol{A} = \boldsymbol{P}\Lambda\boldsymbol{P}^{-1},$$

one has that

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}^{-\mathrm{T}} = \boldsymbol{P}^{-\mathrm{T}}\Lambda^{\mathrm{T}},$$

where $\boldsymbol{B}^{-\mathrm{T}} := (\boldsymbol{B}^{-1})^{\mathrm{T}}$. In the original variables equation (2.24) can then be rewritten as

$$\boldsymbol{P}^{-\mathrm{T}}\boldsymbol{e}_j \cdot \int_0^{2\pi} \mathrm{e}^{-\boldsymbol{A}s}\boldsymbol{b}(s)\,\mathrm{d}s = 0, \quad j = 1, 2. \tag{2.25}$$

A geometric interpretation of equation (2.25) is that the right-hand side of equation (2.22) is orthogonal to $\mathrm{span}\{\boldsymbol{P}^{-\mathrm{T}}\boldsymbol{e}_1, \boldsymbol{P}^{-\mathrm{T}}\boldsymbol{e}_2\}$, i.e., the eigenspace of $\boldsymbol{A}^{\mathrm{T}}$ associated with the eigenvalues $\pm\mathrm{i}$.

Example. Consider

$$\ddot{x} + \omega^2 x = f(t), \quad f(t+2\pi) = f(t).$$

By Corollary 2.47 there is a unique 2π-periodic solution if

$$\omega \neq \ell, \quad \ell \in \mathbb{N}_0.$$

Now suppose that the forcing is resonant, i.e., $\omega \in \mathbb{N}_0$. The right-hand side of equation (2.22) is given by

$$\frac{1}{\omega}\int_0^{2\pi} f(s) \begin{pmatrix} \sin(\omega s) \\ \omega\cos(\omega s) \end{pmatrix} \mathrm{d}s;$$

hence, upon applying equation (2.25) there will exist a 2π-periodic solution if and only if

$$\int_0^{2\pi} f(s)\sin(\omega s)\,\mathrm{d}s = \int_0^{2\pi} f(s)\cos(\omega s)\,\mathrm{d}s = 0. \tag{2.26}$$

Otherwise, the resonant forcing will produce unbounded growth. Note that if one writes $f(t)$ in a Fourier series, i.e.,

$$f(t) = f_0 + \sum_{n=1}^{\infty} a_n\cos(nt) + \sum_{n=1}^{\infty} b_n\sin(nt),$$

then equation (2.26) is equivalent to requiring that $a_1 = b_1 = 0$.

2.3.2. Example: the forced linear Schrödinger equation

Consider

$$\mathrm{i}q_t + \frac{1}{2}q_{xx} - \omega q = \epsilon p(t)q, \quad p(t+T) = p(t),$$

with the boundary condition $q(-N\pi, t) = q(N\pi, t)$ for some $N \in \mathbb{N}$ and for all $t \geq 0$. Upon using a Fourier decomposition and setting

$$q(x,t) = \sum_{n=-\infty}^{+\infty} q_n(t)\mathrm{e}^{\mathrm{i}nx/N},$$

one sees that for each $n \in \mathbb{Z}$,

$$\mathrm{i}\dot{q}_n - \alpha_n q_n = \epsilon p(t)q_n, \quad \alpha_n := \omega + \frac{n^2}{N^2}.$$

Upon writing $q_n := u_n + \mathrm{i}v_n$ one then gets the ODE

$$\dot{\boldsymbol{x}}_n = \boldsymbol{A}_n(t)\boldsymbol{x}_n, \quad \boldsymbol{A}_n(t) := \boldsymbol{J}\boldsymbol{H}(t),$$

where

$$\boldsymbol{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \boldsymbol{H}(t) := -(\alpha_n + \epsilon p(t))\mathbb{1}.$$

One can understand the dynamical behavior of $\boldsymbol{x}_n(t)$ through the use of Floquet theory. Set

$$\beta_n(t) := \alpha_n t + \epsilon\int_0^t p(s)\,\mathrm{d}s.$$

Since $\int \boldsymbol{A}_n(t)\,\mathrm{d}t \cdot \boldsymbol{A}_n(t) = \boldsymbol{A}_n(t)\cdot\int\boldsymbol{A}_n(t)\,\mathrm{d}t$, as a consequence of Lemma 2.9(b) a fundamental matrix solution is given by

$$\Phi_n(t) = \mathrm{e}^{\int_0^s \boldsymbol{A}_n(s)\,\mathrm{d}s} = \begin{pmatrix} \cos\beta_n(t) & \sin\beta_n(t) \\ -\sin\beta_n(t) & \cos\beta_n(t) \end{pmatrix}.$$

The Floquet multipliers $\lambda_n^{\pm}$ are the eigenvalues of $\Phi_n(T)$, and are easily seen to be given by

$$\lambda_n^{\pm} = \cos\beta_n(T) \pm \mathrm{i}\sin\beta_n(T) = \mathrm{e}^{\mathrm{i}\theta_n^{\pm}(T)},$$

where

$$\theta_n^{\pm}(T) = \pm(\alpha_n + \epsilon\bar{p})T, \quad \bar{p} := \frac{1}{T}\int_0^T p(s)\,\mathrm{d}s.$$

The two linearly independent solutions are given by

$$\boldsymbol{x}_n^{\pm}(t) = \boldsymbol{p}_n^{\pm}(t)\mathrm{e}^{\mathrm{i}\theta_n^{\pm}(t)}, \quad \boldsymbol{p}_n^{\pm}(t) = \boldsymbol{p}_n^{\pm}(t+T);$$

hence, one has that $\boldsymbol{x}_n(t)$ will be ℓT-periodic if

$$\alpha_n + \epsilon\bar{p} = \frac{2\pi}{T}\frac{j}{\ell},$$

where $j, \ell \in \mathbb{Z}$ are relatively prime; otherwise, the motion will be bounded but quasi-periodic. Note that since α_n contains the free parameter ω, one can always guarantee that at least one of the Fourier coefficients will be periodic. The full solution to the linear problem will be bounded but quasi-periodic.

Remark 2.49. It is an interesting exercise to attempt to solve the linear problem

$$\mathrm{i}q_t + \frac{1}{2}q_{xx} - \omega q = \epsilon p(t)\cos x\, q, \quad q(-N\pi, t) = q(N\pi, t).$$

What restrictions could you make to make the problem more tractable?

2.3.3. Example: linear Hamiltonian systems

In many applications systems of the type

$$\dot{\boldsymbol{x}} = \boldsymbol{J}\boldsymbol{H}(t)\boldsymbol{x}, \quad \boldsymbol{H}(t+T) = \boldsymbol{H}(t) \tag{2.27}$$

arise, where $\boldsymbol{H}(t) \in \mathbb{R}^{2n\times 2n}$ is symmetric and $\boldsymbol{J}$ is nonsingular and skew-symmetric, i.e., $\boldsymbol{J}^{\mathrm{T}} = -\boldsymbol{J}$. In applications one often also has that $\boldsymbol{J}^{-1} = \boldsymbol{J}^{\mathrm{T}}$, so that $\boldsymbol{J}\boldsymbol{J} = -\mathbb{1}$. One such example was given in Section 2.3.2.

Let $\Phi(t)$ represent the principal fundamental matrix solution to equation (2.27). Since $(\boldsymbol{J}\boldsymbol{H}(t))^{\mathrm{T}} = -\boldsymbol{H}(t)\boldsymbol{J}$, the adjoint problem associated with equation (2.27) is given by

$$\dot{\boldsymbol{y}} = \boldsymbol{H}(t)\boldsymbol{J}\boldsymbol{y}. \tag{2.28}$$

Assuming that $\boldsymbol{J}^2 = -\mathbb{1}$, one gets that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{J}\boldsymbol{x} &= -\boldsymbol{J}^2\boldsymbol{H}(t)\boldsymbol{J}(\boldsymbol{J}\boldsymbol{x})\\ &= \boldsymbol{H}(t)\boldsymbol{J}(\boldsymbol{J}\boldsymbol{x});\end{aligned}$$

hence, if $\boldsymbol{x}$ solves equation (2.27), then $\boldsymbol{J}\boldsymbol{x}$ solves the adjoint problem equation (2.28). Thus, $\boldsymbol{J}\Phi$ is a solution to the adjoint problem, so that the principal fundamental matrix solution to equation (2.28) is given by

$$\Psi(t) = -\boldsymbol{J}\Phi(t)\boldsymbol{J}.$$

Now, it can be checked that another solution to equation (2.28) is $\Phi(t)^{-\mathrm{T}}$. Uniqueness then implies that

$$\Phi(t)^{-1} = -\boldsymbol{J}\Phi(t)^{\mathrm{T}}\boldsymbol{J} = \boldsymbol{J}\Phi(t)^{\mathrm{T}}\boldsymbol{J}^{-1},$$

i.e., Φ^{T} is similar to Φ^{-1}. Thus, for $\mu \in \sigma(\Phi(T))$ one has that $\mu^{-1} \in \sigma(\Phi(T))$. Since $\Phi(T) \in \mathbb{R}^{2n\times 2n}$, one also has that if $\mu \in \sigma(\Phi(T))$, then $\mu^* \in \sigma(\Phi(T))$. This argument yields the following lemma:

Lemma 2.50. *Consider equation (2.27), and suppose that $\boldsymbol{J} \in \mathbb{R}^{2n\times 2n}$ is skew-symmetric with $\boldsymbol{J}^{-1} = \boldsymbol{J}^{\mathrm{T}}$. If $\Phi(t)$ is the principle fundamental matrix solution, then $\mu \in \sigma(\Phi(T))$ implies that $1/\mu, \mu^*, 1/\mu^*, \in \sigma(\Phi(T))$.*

Now suppose that

$$\boldsymbol{H}(t)=\boldsymbol{A}+\epsilon\boldsymbol{P}(t),\quad \boldsymbol{P}(t+T)=\boldsymbol{P}(t),$$

where both $\boldsymbol{A}$ and $\boldsymbol{P}(t)$ are symmetric. By following the argument leading to Lemma 2.50 it can be seen that if $\lambda\in\sigma(\boldsymbol{JA})$, then $-\lambda,\pm\lambda^*\in\sigma(\boldsymbol{JA})$. If one assumes that $\sigma(\boldsymbol{A})=\sigma^{\mathrm{s}}(\boldsymbol{A})$ or $\sigma(\boldsymbol{A})=\sigma^{\mathrm{u}}(\boldsymbol{A})$, then it can be shown that one actually has $\sigma(\boldsymbol{JA})\subset\mathrm{i}\mathbb{R}$ [**11**, **12**]. Under this assumption, let $\pm\mathrm{i}\mu_j\in\sigma(\boldsymbol{JA})$ for $j=1,\dots,n$. When $\epsilon=0$, the characteristic multipliers are given by

$$\rho_j^{\pm}=\mathrm{e}^{\pm\mathrm{i}\mu_jT},\quad j=1,\dots,n.$$

These multipliers satisfy $|\rho_j^{\pm}|=1$, and are distinct if

$$\mu_j\neq 0\pmod{\frac{\pi}{T}},\quad j=1,\dots,n;\quad \mu_j\pm\mu_k\neq 0\pmod{\frac{2\pi}{T}},\quad j\neq k. \tag{2.29}$$

Since the multipliers vary continuously under perturbation, if equation (2.29) holds when $\epsilon=0$ one has by Lemma 2.50 that there is an $\epsilon_0>0$ such that for the perturbed problem the multipliers are simple and satisfy $|\rho|=1$ for $0\leq\epsilon<\epsilon_0$. Hence, as a consequence of Lemma 2.44 one has that the trivial solution is stable for the perturbed problem.

Example. Consider the following variation of the problem given in Section 2.3.2:

$$\mathrm{i}q_t+\frac{1}{2}q_{xx}-\omega q=2\epsilon p(t)\cos(x)q,\quad p(t+T)=p(t)$$

with the boundary condition $q(-N\pi,t)=q(N\pi,t)$ for some $N\in\mathbb{N}$ and for all $t\geq 0$. Upon setting

$$q(x,t)=\sum_{n=-\infty}^{+\infty}q_n(t)\mathrm{e}^{\mathrm{i}nx/N},$$

and using the fact that

$$\cos(x)=\frac{1}{2}(\mathrm{e}^{\mathrm{i}x}+\mathrm{e}^{-\mathrm{i}x}),$$

one sees that

$$\cos(x)q=\frac{1}{2}\sum_{n=-\infty}^{+\infty}(q_{n-N}+q_{n+N})\mathrm{e}^{\mathrm{i}nx/N}.$$

Hence, for each $n\in\mathbb{Z}$,

$$\mathrm{i}\dot{q}_n-\alpha_nq_n=\epsilon p(t)(q_{n-N}+q_{n+N}),\quad \alpha_n:=\omega+\frac{n^2}{2N^2}.$$

Upon writing $q_n:=u_n+\mathrm{i}v_n$, and setting

$$\boldsymbol{J}:=\begin{pmatrix}0&1\\-1&0\end{pmatrix},$$

one gets for $\boldsymbol{x}_n:=(u_n,v_n)^{\mathrm{T}}$ the ODE

$$\dot{\boldsymbol{x}}_n=-\boldsymbol{J}(\alpha_n\mathbb{1}\boldsymbol{x}_n+\epsilon p(t)(\boldsymbol{x}_{n-N}+\boldsymbol{x}_{n+N})). \tag{2.30}$$

For fixed n, set

$$\boldsymbol{y}_j:=\boldsymbol{x}_{n+jN},\quad \beta_j:=\alpha_{n+jN}=\alpha_n+2\frac{j}{N}+j^2.$$

The system equation (2.30) can then be rewritten as

$$\dot{\boldsymbol{y}}_j=-\boldsymbol{J}(\alpha_j\mathbb{1}\boldsymbol{y}_j+\epsilon p(t)(\boldsymbol{y}_{j-1}+\boldsymbol{y}_{j+1})). \tag{2.31}$$

Since $j\in\mathbb{Z}$, at this point equation (2.31) is an infinite-dimensional ODE. Now truncate by supposing that for some $M\geq 1$ one has that $\boldsymbol{y}_{\pm k}(t)\equiv\boldsymbol{0}$, $k\geq M+1$. Under this restriction equation (2.31) then becomes $4(M-1)$ dimensional, and can be written as

$$\dot{\boldsymbol{y}}=\mathbb{J}(\boldsymbol{D}+\epsilon p(t)\boldsymbol{B})\boldsymbol{y}, \tag{2.32}$$

where
$$\boldsymbol{y} := (\boldsymbol{y}_{-M}, \dots, \boldsymbol{y}_M)^{\mathrm{T}}, \quad \mathbb{J} := \operatorname{diag}(\boldsymbol{J}, \dots, \boldsymbol{J}), \quad \boldsymbol{D} := -\operatorname{diag}(\beta_{-M}\mathbb{1}, \dots, \beta_M\mathbb{1}),$$
and $\boldsymbol{B}$ is symmetric and satisfies $\boldsymbol{B}_{i,i\pm1} = -\mathbb{1}$ and is zero elsewhere (with an obvious abuse of notation). When $\epsilon = 0$, one has that
$$\pm\mathrm{i}\beta_j \in \sigma(\mathbb{J}\boldsymbol{D}), \quad j = -M, \dots, M.$$
Using equation (2.29) and applying the theory preceding this example it is then known that all solutions will be bounded for $0 \le \epsilon \ll 1$ if
$$\beta_j \ne 0 \pmod{\frac{\pi}{T}}, \quad j = -M, \dots, M; \quad \beta_j \pm \beta_k \ne 0 \pmod{\frac{2\pi}{T}}, \quad j \ne k.$$
It is an exercise to give precise conditions on ω, n, N such that the above holds true.

2.3.4. Example: Hill's equation

Herein we will consider a simple example problem which is surprisingly difficult to analyze (e.g., see [**14**]). Consider
$$\ddot{x} + a(t)x = 0, \tag{2.33}$$
where $a : \mathbb{R} \mapsto \mathbb{R}$ is a continuous T-periodic function. A simple rescaling argument yields that without loss of generality one may assume that $T = \pi$. Herein the focus will solely be on developing a stability and instability criterion. It will first be shown that if
$$0 \le \int_0^\pi a(s)\,\mathrm{d}s \le \frac{4}{\pi}, \quad a(t) \ge 0, \tag{2.34}$$
then the trivial solution is stable. In other words, it will be shown that equation (2.34) yields that the Floquet multipliers associated with equation (2.33) have modulus equal to unity (see Lemma 2.44). Note that after writing equation (2.33) as the first-order system $\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}$ with
$$\boldsymbol{A}(t) := \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix}.$$
one has that trace $\boldsymbol{A}(t) = 0$; hence, by Lemma 2.45 the Floquet multipliers satisfy
$$\lambda_1\lambda_2 = 1. \tag{2.35}$$

If $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \ne \lambda_2$, then by equation (2.35) one has that without loss of generality $|\lambda_1| > 1$, so that the trivial solution is unstable. If $\lambda_1 = \lambda_2 = 1$, then there exists a solution x_{p} such that $x_{\mathrm{p}}(t+\pi) = x_{\mathrm{p}}(t)$, whereas if $\lambda_1 = \lambda_2 = -1$, then there exists a solution x_{p} such that $x_{\mathrm{p}}(t+2\pi) = x_{\mathrm{p}}(t)$. In either case, by using reduction of order a second linearly independent solution is given by $x_2(t) = u(t)x_{\mathrm{p}}(t)$, where
$$x_{\mathrm{p}}\ddot{u} + 2\dot{x}_{\mathrm{p}}\dot{u} = 0.$$
This solution may or may not be unbounded as $t \to +\infty$. In conclusion, if $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \ne \lambda_2$, the trivial solution is unstable. Conversely, if $\lambda_1, \lambda_2 \notin \mathbb{R}$, then $\lambda_2 = \lambda_1^*$ with $|\lambda_1| = 1$, so that all solutions are bounded for all $t \in \mathbb{R}$.

Now suppose that $\lambda_1, \lambda_2 \in \mathbb{R}$. There exists a solution $x(t)$ such that $x(t+\pi) = \lambda_1 x(t)$. Either $x(t) \ne 0$ for all $t \in \mathbb{R}$, or $x(t) = 0$ has infinitely many solutions, with two consecutive zeros z_1, z_2 satisfying $0 \le z_2 - z_1 \le \pi$. In the first case $x(\pi) = \lambda_1 x(0)$ and $\dot{x}(\pi) = \lambda_1\dot{x}(0)$, so that
$$\frac{\dot{x}(\pi)}{x(\pi)} = \frac{\dot{x}(0)}{x(0)}.$$
Since $x(t)$ solves equation (2.33), upon dividing by x and integrating by parts one gets that
$$\int_0^\pi \frac{\dot{x}(s)^2}{x(s)^2}\,\mathrm{d}s + \int_0^\pi a(s)\,\mathrm{d}s = 0.$$

This is a contradiction, as $a(t) \geq 0$. In the second case, suppose that $x(t) > 0$ for $t \in (z_1, z_2)$. Let

$$x(c) := \max_{t \in [z_1, z_2]} x(t).$$

For any $t_1, t_2 \in (z_1, z_2)$ the hypothesis on $a(t)$ implies that

$$\frac{4}{\pi} \geq \int_0^\pi a(s)\,\mathrm{d}s \geq \int_{t_1}^{t_2} \frac{|\ddot{x}(s)|}{x(s)}\,\mathrm{d}s \geq \frac{1}{x(c)} \int_{t_1}^{t_2} |\ddot{x}(s)|\,\mathrm{d}s \geq \frac{1}{x(c)} \left| \int_{t_1}^{t_2} \ddot{x}(s)\,\mathrm{d}s \right| = \frac{|\dot{x}(t_2) - \dot{x}(t_1)|}{x(c)}.$$

By the Mean Value Theorem there exists specific $t_1, t_2 \in (z_1, z_2)$ such that

$$\dot{x}(t_1) = \frac{x(c) - x(z_1)}{c - z_1}, \quad \dot{x}(t_2) = \frac{x(z_2) - x(c)}{z_2 - c}.$$

Since $x(z_1) = x(z_2) = 0$, this yields

$$\dot{x}(t_2) - \dot{x}(t_1) = x(c) \frac{z_2 - z_1}{(c - z_1)(z_2 - c)} = x(c) \left(\frac{1}{c - z_1} + \frac{1}{z_2 - c} \right) > x(c) \frac{4}{z_2 - z_1}.$$

Thus,

$$\frac{4}{\pi} \geq \int_0^\pi a(s)\,\mathrm{d}s > \frac{4}{z_2 - z_1} \geq \frac{4}{\pi},$$

which yields a contradiction. Hence, $\lambda_1, \lambda_2 \notin \mathbb{R}$.

Remark 2.51. Equation (2.34) is by no means necessary. If one sets $a(t) \equiv a_0^2$ with $a_0 \in \mathbb{R}^+$, then equation (2.34) becomes $a_0 \leq 2/\pi$. However, in this case it is known that the trivial solution is stable for any value of a_0. Note that the Floquet multipliers in this case are $\mathrm{e}^{\pm \mathrm{i} a_0 \pi}$, and are real-valued and equal to unity for $a_0 = 2\ell$, $\ell \in \mathbb{N}$.

Remark 2.52. The restriction on $a(t)$ given in equation (2.34) can be relaxed to

$$\int_0^\pi a(s)\,\mathrm{d}s \geq 0, \quad \int_0^\pi |a(s)|\,\mathrm{d}s \leq \frac{4}{\pi}$$

[**6**, Theorem III.8.2].

The following general result is useful in many applications (e.g., see [**6**, Chapter III.8] and [**14**]).

Lemma 2.53. *Let $\Phi(t)$ represent the principal fundamental matrix solution to equation (2.33). If $|\operatorname{trace}(\Phi(\pi))| < 2$, then $x = 0$ is stable, whereas if $|\operatorname{trace}(\Phi(\pi))| > 2$, the solution $x = 0$ is unstable.*

Proof: The Floquet multipliers satisfy equation (2.35), i.e.,

$$\det(\Phi(\pi)) = \lambda_1 \lambda_2 = 1;$$

furthermore, by Lemma 2.45 $\operatorname{trace}(\Phi(\pi)) = \lambda_1 + \lambda_2$. This yields that

$$\lambda_{1,2} = \frac{1}{2}(\operatorname{trace}(\Phi(\pi)) \pm \sqrt{(\operatorname{trace}(\Phi(\pi))^2 - 4}).$$

If $\operatorname{trace}(\Phi(\pi)) < 2$, then $\lambda_1 \neq \lambda_2$ with $|\lambda_j| = 1$ for $j = 1, 2$; hence, $x = 0$ is stable. If $\operatorname{trace}(\Phi(\pi)) > 2$, then $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 > 1$, which implies that $x = 0$ is unstable. $\square$

If one denotes the principal fundamental matrix solution via $\Phi(t) = (\boldsymbol{x}(t)\, \boldsymbol{y}(t))$, then

$$\operatorname{trace}(\Phi(\pi)) = x_1(\pi) + \dot{y}_1(\pi);$$

hence, one can paraphrase Lemma 2.53 to say that if

$$x_1(\pi) + \dot{y}_1(\pi) > 2, \tag{2.36}$$

then the trivial solution is unstable.

3. The manifold theorems

Consider the two systems

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \tag{3.1}$$

and

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{r}(\boldsymbol{x}), \tag{3.2}$$

where $|\boldsymbol{r}(\boldsymbol{x})| = \mathcal{O}(|\boldsymbol{x}|^2)$. As a consequence of Theorem 2.29 the behavior of the flow associated with equation (3.1) is completely understood. The stability results in Corollary 2.37 and Corollary 2.38 state that the solution behavior for these systems is asymptotically equivalent if $\sigma^{\mathrm{c}}(\boldsymbol{A}) = \varnothing$ with the additional condition that either $\sigma^{\mathrm{s}}(\boldsymbol{A}) = \varnothing$ or $\sigma^{\mathrm{u}}(\boldsymbol{A}) = \varnothing$. What if the second addition is not the case?

The first goal herein is to show that as long as $\sigma^{\mathrm{c}}(\boldsymbol{A}) = \varnothing$, then the flow associated with equation (3.2) is qualitatively similar to that for equation (3.1). In particular, this will imply that if $\sigma^{\mathrm{u}}(\boldsymbol{A}) \neq \varnothing$, then the solution $\boldsymbol{x} = \boldsymbol{0}$ to equation (3.2) is unstable.

As seen in the discussion leading to equation (4.3) it can be assumed that

$$\boldsymbol{A} = \mathrm{diag}(\boldsymbol{A}^{\mathrm{s}}, \boldsymbol{A}^{\mathrm{u}}),$$

where $\boldsymbol{A}^{\mathrm{s,u}} \in \mathbb{R}^{(n_{\mathrm{s}},n_{\mathrm{u}})\times(n_{\mathrm{s}},n_{\mathrm{u}})}$ with $n_{\mathrm{s}} + n_{\mathrm{u}} = n$, and $\sigma(\boldsymbol{A}^{\mathrm{s,u}}) = \sigma^{\mathrm{s,u}}(\boldsymbol{A}^{\mathrm{s,u}})$. Define the projection operators $\Pi_{\mathrm{s,u}}$ by

$$\Pi_{\mathrm{s}} := \mathrm{diag}(\mathbb{1}_{\mathrm{s}}, \boldsymbol{0}), \quad \Pi_{\mathrm{u}} := \mathrm{diag}(\boldsymbol{0}, \mathbb{1}_{\mathrm{u}}),$$

where $\mathbb{1}_{\mathrm{s,u}} \in \mathbb{R}^{(n_{\mathrm{s}},n_{\mathrm{u}})\times(n_{\mathrm{s}},n_{\mathrm{u}})}$. Note that the projection operators satisfy the properties

$$\Pi_{\mathrm{s}}\Pi_{\mathrm{u}} = \Pi_{\mathrm{u}}\Pi_{\mathrm{s}} = \boldsymbol{0}, \quad \Pi_{\mathrm{s,u}}^2 = \Pi_{\mathrm{s,u}}, \quad \Pi_{\mathrm{s}} + \Pi_{\mathrm{u}} = \mathbb{1}; \tag{3.3}$$

furthermore,

$$\Pi_{\mathrm{s,u}}\mathrm{e}^{\boldsymbol{A}t} = \mathrm{e}^{\boldsymbol{A}t}\Pi_{\mathrm{s,u}}. \tag{3.4}$$

As a consequence of Theorem 2.29 one has that there exists a $C, \alpha, \beta \in \mathbb{R}^+$ such that

$$|\mathrm{e}^{\boldsymbol{A}t}\Pi_{\mathrm{s}}| \le C\mathrm{e}^{-\alpha t},\ t \ge 0; \quad |\mathrm{e}^{\boldsymbol{A}t}\Pi_{\mathrm{u}}| \le C\mathrm{e}^{\beta t},\ t \le 0. \tag{3.5}$$

Let $\epsilon_0 \in \mathbb{R}^+$ be such that for $0 < \epsilon \le \epsilon_0$,

$$C\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\epsilon < 1, \tag{3.6}$$

and let $\delta_1 \in \mathbb{R}^+$ be such that for $|\boldsymbol{x}| \le \delta_1$, $|\boldsymbol{r}(\boldsymbol{x})| \le \epsilon_0|\boldsymbol{x}|$. Since $\boldsymbol{r}(\boldsymbol{x})$ is smooth and satisfies the estimate $|\boldsymbol{r}(\boldsymbol{x})| = \mathcal{O}(|\boldsymbol{x}|^2)$, for each given $\eta \in \mathbb{R}^+$ sufficiently small there exists a $\delta_\eta \in \mathbb{R}^+$ such that if $|\boldsymbol{x}_1|, \boldsymbol{x}_2| \le \delta_\eta$, then

$$|\boldsymbol{r}(\boldsymbol{x}_2) - \boldsymbol{r}(\boldsymbol{x}_1)| \le \eta|\boldsymbol{x}_2 - \boldsymbol{x}_1|.$$

Let $\delta_2 \in \mathbb{R}^+$ be such that if $|\boldsymbol{x}_1|, |\boldsymbol{x}_2| \le \delta_2$, then

$$|\boldsymbol{r}(\boldsymbol{x}_2) - \boldsymbol{r}(\boldsymbol{x}_1)| \le \epsilon_0|\boldsymbol{x}_2 - \boldsymbol{x}_1|. \tag{3.7}$$

Set

$$X := C^0([0, +\infty); \mathbb{R}^n), \quad \|\boldsymbol{x}\| := \sup_{t\ge 0}|\boldsymbol{x}(t)|,$$

and for $\delta_0 := \min\{\delta_1, \delta_2\}$,

$$D := \{\boldsymbol{x}(t) \in X \,:\, \|\boldsymbol{x}\| \le \delta_0\}.$$

Now let $\boldsymbol{x}_0 \in \mathbb{R}^n$ be given so that $\Pi_{\mathrm{s}}\boldsymbol{x}_0 = \boldsymbol{x}_0$; furthermore, suppose that

$$|\boldsymbol{x}_0| \le \frac{1}{C}\left[1 - C\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\epsilon_0\right]\delta_0. \tag{3.8}$$

Note that for equation (3.1) the resulting solution satisfies

$$|\boldsymbol{x}(t)| \le C\mathrm{e}^{-\alpha t}|\boldsymbol{x}_0| \le \left[1 - C\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\epsilon_0\right]\delta_0. \tag{3.9}$$

The second inequality follows from equation (3.8). For such an $\boldsymbol{x}_0$, define the mapping $\mathcal{T} : X \mapsto X$ by

$$\mathcal{T}\boldsymbol{y} := \boldsymbol{x}(t) + \int_0^t \mathrm{e}^{\boldsymbol{A}(t-s)}\Pi_{\mathrm{s}}\boldsymbol{r}(\boldsymbol{y}(s))\,\mathrm{d}s - \int_t^{+\infty} \mathrm{e}^{\boldsymbol{A}(t-s)}\Pi_{\mathrm{u}}\boldsymbol{r}(\boldsymbol{y}(s))\,\mathrm{d}s. \tag{3.10}$$

The last integral is well-defined as a consequence of equation (3.5). If $\boldsymbol{y} \in D$, then one has that

$$|\mathcal{T}\boldsymbol{y}| \le C\mathrm{e}^{-\alpha t}|\boldsymbol{x}_0| + C\epsilon_0\|\boldsymbol{y}\|\int_0^t \mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}s + C\epsilon_0\|\boldsymbol{y}\|\int_t^{+\infty}\mathrm{e}^{\beta(t-s)}\,\mathrm{d}s,$$

i.e.,

$$\|\mathcal{T}\boldsymbol{y}\| \le C|\boldsymbol{x}_0| + C\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\epsilon_0\delta_0.$$

As a consequence of equation (3.9) one then has that $\mathcal{T} : D \mapsto D$. Upon using equation (3.7) one further has that for $\boldsymbol{y}_1, \boldsymbol{y}_2 \in D$,

$$\|\mathcal{T}\boldsymbol{y}_2 - \mathcal{T}\boldsymbol{y}_1\| \le C\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\epsilon_0\|\boldsymbol{y}_2 - \boldsymbol{y}_1\|.$$

Hence, by equation (3.6) one has that $\mathcal{T} : D \mapsto D$ is a contraction map, so that by Theorem 2.34 the mapping has a unique fixed point $\boldsymbol{y}_{\mathrm{s}} \in D$.

Differentiating with respect to t and using equation (3.3) yields that $\boldsymbol{y}_{\mathrm{s}}$ is a bounded solution to equation (3.2) with the initial condition

$$\boldsymbol{y}_{\mathrm{s}}(0) = \boldsymbol{x}_0 - \int_0^{+\infty}\mathrm{e}^{-\boldsymbol{A}s}\Pi_{\mathrm{u}}\boldsymbol{r}(\boldsymbol{y}_{\mathrm{s}}(s))\,\mathrm{d}s.$$

Thus, for each bounded solution to equation (3.1) there exists a corresponding unique bounded solution to equation (3.2). Note that

$$\Pi_{\mathrm{s}}\boldsymbol{y}_{\mathrm{s}}(0) = \boldsymbol{x}_0, \quad \Pi_{\mathrm{u}}\boldsymbol{y}_{\mathrm{s}}(0) = -\int_0^{+\infty}\mathrm{e}^{-\boldsymbol{A}s}\Pi_{\mathrm{u}}\boldsymbol{r}(\boldsymbol{y}_{\mathrm{s}}(s))\,\mathrm{d}s;$$

hence, there exists an $h^{\mathrm{s}} : E^{\mathrm{s}} \mapsto E^{\mathrm{u}}$ given by

$$h^{\mathrm{s}}(\boldsymbol{x}_0) := -\int_0^{+\infty}\mathrm{e}^{-\boldsymbol{A}s}\Pi_{\mathrm{u}}\boldsymbol{r}(\boldsymbol{y}_{\mathrm{s}}(s))\,\mathrm{d}s,$$

such that for the initial condition $\boldsymbol{y}(0) = \boldsymbol{x}_0 + h^{\mathrm{s}}(\boldsymbol{x}_0)$ one has a bounded solution to equation (3.2). It is a nontrivial exercise to show that if $\boldsymbol{r}(\boldsymbol{x})$ is C^r for some $r \in \mathbb{N}$, then h^{s} is C^{r-1}.

Now let $0 < \tilde{\alpha} < \alpha$ be given, and It is clear that

$$|\boldsymbol{r}(\boldsymbol{x})| \le K|\boldsymbol{x}|^2 \le K\mathrm{e}^{\tilde{\alpha}t}|\boldsymbol{x}|^2,$$

so that for the fixed point $\boldsymbol{y}_{\mathrm{s}}$ to equation (3.10) one has the estimate

$$\mathrm{e}^{\tilde{\alpha}t}|\boldsymbol{y}_{\mathrm{s}}(t)| \le C\mathrm{e}^{-(\alpha-\tilde{\alpha})t}|\boldsymbol{x}_0| + CK\int_0^t \mathrm{e}^{-(\alpha-\tilde{\alpha})(t-s)}\mathrm{e}^{2\tilde{\alpha}s}|\boldsymbol{y}_{\mathrm{s}}(s)|^2\,\mathrm{d}s + CK\int_t^{+\infty}\mathrm{e}^{(\beta+\tilde{\alpha})(t-s)}\mathrm{e}^{2\tilde{\alpha}s}|\boldsymbol{y}_{\mathrm{s}}(s)|^2\,\mathrm{d}s.$$

If one defines the norm

$$\|\boldsymbol{x}\|_{\mathrm{w}} := \sup_{t\ge 0}\mathrm{e}^{\tilde{\alpha}t}|\boldsymbol{x}(t)|,$$

then from the above one gets that

$$\|\boldsymbol{y}_{\mathrm{s}}\|_{\mathrm{w}} \le C|\boldsymbol{x}_0| + CK\left(\frac{1}{\alpha-\tilde{\alpha}}+\frac{1}{\beta+\tilde{\alpha}}\right)\|\boldsymbol{y}_{\mathrm{s}}\|_{\mathrm{w}}^2.$$

Hence, if $|\boldsymbol{y}_\mathrm{s}(0)| < C|\boldsymbol{x}_0| + \mathcal{O}(|\boldsymbol{x}_0|^2)$, which is possible since $|h^\mathrm{s}(\boldsymbol{x}_0)| = \epsilon_0\mathcal{O}(\delta_0)$ and $|\boldsymbol{x}_0| = \mathcal{O}(\delta_0)$, one has that $\|\boldsymbol{y}_\mathrm{s}\|_\mathrm{w} \le C|\boldsymbol{x}_0|$. This implies that

$$|\boldsymbol{y}_\mathrm{s}(t)| \le C|\boldsymbol{x}_0|\mathrm{e}^{-\tilde{\alpha}t}. \tag{3.11}$$

Upon using equation (3.3) and equation (3.4) one further has that

$$\Pi_\mathrm{u}\boldsymbol{y}_\mathrm{s}(t) = -\int_t^{+\infty} \mathrm{e}^{\boldsymbol{A}(t-s)}\Pi_\mathrm{u}\boldsymbol{r}(\boldsymbol{y}_\mathrm{s}(s))\,\mathrm{d}s,$$

so that by using equation (3.11) one can conclude that

$$\lim_{t\to+\infty} |\Pi_\mathrm{u}\boldsymbol{y}_\mathrm{s}(t)| = 0.$$

Thus, the bounded solutions decay exponentially fast as $t \to +\infty$; furthermore, they approach E^s in the limit.

Remark 3.1. There is an analogous result for $t \in \mathbb{R}^-$; in particular, the bounded solutions for $t \le 0$ decay exponentially fast and approach E^u as $t \to -\infty$.

For equation (3.1) one has the existence of invariant subspaces on which the behavior of the flow is completely characterized (see Theorem 2.29). One cannot expect invariant subspaces for the nonlinear system of equation (3.2); however, perhaps one can expect invariant *surfaces* which are realized as a smooth deformation of a subspace.

Definition 3.2. A space X is a topological manifold of dimension k if each point $x \in X$ has a neighborhood homeomorphic to the unit ball in $\mathbb{R}^k$. In particular, the graphs of smooth functions are manifolds.

Armed with this definition and the above discussion, we are now able to state the manifold theorems for equation (3.2). The proofs of these theorems in the case that $\sigma^\mathrm{c}(\boldsymbol{A}) \ne \varnothing$, as well as the implications of the existence of the center-manifold W^c, will be given at a later time.

Definition 3.3. Let N be a given small neighborhood of $\boldsymbol{x} = \boldsymbol{0}$. The stable manifold, W^s, is

$$W^\mathrm{s} := \{\boldsymbol{x}_0 \in N \,:\, \phi_t(\boldsymbol{x}_0) \in N\,\forall\, t \ge 0 \text{ and } \phi_t(\boldsymbol{x}_0) \to \boldsymbol{0} \text{ exponentially fast as } t \to +\infty\}.$$

The unstable manifold, W^u, is

$$W^\mathrm{u} := \{\boldsymbol{x}_0 \in N \,:\, \phi_t(\boldsymbol{x}_0) \in N\,\forall\, t \le 0 \text{ and } \phi_t(\boldsymbol{x}_0) \to 0 \text{ exponentially fast as } t \to -\infty\}.$$

The center manifold, W^c, is invariant relative to N, i.e., if $\boldsymbol{x}_0 \in W^\mathrm{c}$, then $\phi_t(\boldsymbol{x}_0) \in W^\mathrm{c} \cap N$ for all $t \in \mathbb{R}$. Furthermore, $W^\mathrm{c} \cap W^\mathrm{s} = W^\mathrm{c} \cap W^\mathrm{u} = \{\boldsymbol{0}\}$.

As already stated, as a consequence of Theorem 2.29 one knows that the above manifolds exist for linear systems; furthermore, the manifolds in this case are actually linear subspaces. The below results show that the inclusion of the nonlinear term $\boldsymbol{r}(\boldsymbol{x})$ only serves to "bend" these linear subspaces into smooth surfaces which are tangent to the subspace at the critical point.

Theorem 3.4 (Stable manifold theorem)**.** *There is a neighborhood N of $\boldsymbol{x} = \boldsymbol{0}$ and a C^{r-1} function $h^\mathrm{s} : N \cap E^\mathrm{s} \mapsto E^\mathrm{c} \oplus E^\mathrm{u}$ such that $W^s = \mathrm{graph}(h^\mathrm{s})$.*

Theorem 3.5 (Unstable manifold theorem)**.** *There is a neighborhood N of $\boldsymbol{x} = \boldsymbol{0}$ and a C^{r-1} function $h^\mathrm{u} : N \cap E^\mathrm{u} \mapsto E^\mathrm{s} \oplus E^\mathrm{c}$ such that $W^u = \mathrm{graph}(h^\mathrm{u})$.*

Theorem 3.6 (Center manifold theorem)**.** *There is a neighborhood N of $\boldsymbol{x} = \boldsymbol{0}$ and a C^{r-1} function $h^\mathrm{c} : N \cap E^\mathrm{c} \mapsto E^\mathrm{u} \oplus E^\mathrm{s}$ such that $\mathrm{graph}(h^\mathrm{u})$ is a W^c.*

Remark 3.7. One further has that:

(a) $\dim(W^{\mathrm{s,c,u}}) = \dim(E^{\mathrm{s,c,u}})$

(b) The manifolds are invariant, i.e., if $\boldsymbol{x}_0 \in W^{\mathrm{s,c,u}}$, then $\phi_t(\boldsymbol{x}_0) \in W^{\mathrm{s,c,u}}$ for all $t \in \mathbb{R}$

(c) $W^{\mathrm{s,c,u}}$ is tangent to $E^{\mathrm{s,c,u}}$ at $\boldsymbol{x} = \boldsymbol{0}$

(d) The dynamical behavior on W^s and W^u is determined solely by the linear behavior

(e) W^c is not unique. For example, consider the system

$$\dot{x} = x^2, \quad \dot{y} = -y.$$

4. Stability analysis: the direct method

Consider the autonomous systems

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \tag{4.1}$$

where $\boldsymbol{f} \in C^2(\mathbb{R}^n)$ is such that $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$. The stability results presented in Section 3 rely upon a spectral analysis of $\mathrm{D}\boldsymbol{f}(\boldsymbol{0})$. Even in the case of stability, no indication is given as to how close the initial data must be to the equilibrium solution in order to guarantee stability. Furthermore, the results contained therein leave open the question of stability in the case that $\sigma^{\mathrm{c}}(\boldsymbol{A}) \neq \varnothing$. In this section we will approach the stability question from a different perspective.

4.1. The ω-limit set

Before the dynamics associated with equation (4.1) can be carefully studied, a mathematical description of the associated long-time asymptotics is necessary. In particular, a meaningful way to describe what it means for a time-dependent solution to be stable is necessary. The goal of this subsection is to develop this technology.

Definition 4.1. Let the unique solution to equation (4.1) be denoted by $\phi_t(\boldsymbol{x})$. One says that ϕ_t is the flow defined by the vector field $\boldsymbol{f}(\boldsymbol{x})$.

Restating Theorem 1.16, Theorem 1.17, and Lemma 1.18 in terms of the flow yields:

Lemma 4.2. *The flow associated with equation (4.1) satisfies*

(a) $\phi_0(\boldsymbol{x}) = \boldsymbol{x}$

(b) $\phi_{s+t}(\boldsymbol{x}) = \phi_t[\phi_s(\boldsymbol{x})] = \phi_s[\phi_t(\boldsymbol{x})]$

(c) $\phi_t[\phi_{-t}(\boldsymbol{x})] = \boldsymbol{x}$.

Furthermore, the flow is as smooth as $\boldsymbol{f}(\boldsymbol{x})$.

Lemma 4.3. *If* $\phi_t(\boldsymbol{x}^*) = \boldsymbol{x}^*$ *for all* $t \geq 0$, *then* $\boldsymbol{x}^*$ *is a critical point. If* $\phi_T(\boldsymbol{x}) = \boldsymbol{x}$ *and* $\phi_t(\boldsymbol{x}) \neq \boldsymbol{x}$ *for all* $t \in (0, T)$, *then* $\phi_t(\boldsymbol{x})$ *is a periodic orbit with period* T.

In addition to talking about the flow associated with a single starting value, one can discuss the flow of sets:

Definition 4.4. The flow of a set $K \subset \mathbb{R}^n$ is given by

$$\phi_t(K) := \bigcup_{\boldsymbol{x} \in K} \phi_t(\boldsymbol{x}).$$

Example. If Γ is a periodic orbit, then $\phi_t(\Gamma) = \Gamma$.

Definition 4.5. Let $\boldsymbol{p} \in \mathbb{R}^n$ be given. The positive orbit, $\gamma^+(\boldsymbol{p})$, is given by

$$\gamma^+(\boldsymbol{p}) := \bigcup_{t \geq 0} \phi_t(\boldsymbol{p}),$$

and the negative orbit, $\gamma^-(\boldsymbol{p})$, is given by

$$\gamma^-(\boldsymbol{p}) := \bigcup_{t \leq 0} \phi_t(\boldsymbol{p}),$$

The orbit, $\gamma(\boldsymbol{p})$, is given by $\gamma(\boldsymbol{p}) := \gamma^-(\boldsymbol{p}) \cup \gamma^+(\boldsymbol{p})$.

We are now in position to describe the long-time asymptotics of the flow.

Definition 4.6. The ω-limit set is given by

$$\omega(\boldsymbol{p}) := \bigcap_{\tau\ge0} \overline{\{\phi_t(\boldsymbol{p}) \,:\, t\ge\tau\}},$$

and the α-limit set is given by

$$\alpha(\boldsymbol{p}) := \bigcap_{\tau\le0} \overline{\{\phi_t(\boldsymbol{p}) \,:\, t\le\tau\}}.$$

Remark 4.7. It is an exercise to show that

$$\omega(\boldsymbol{p}) = \{\boldsymbol{y}\in\mathbb{R}^n \,:\, \phi_{t_k}(\boldsymbol{p})\to\boldsymbol{y} \text{ as } t_k\to+\infty\},\quad \alpha(\boldsymbol{p}) = \{\boldsymbol{y}\in\mathbb{R}^n \,:\, \phi_{t_k}(\boldsymbol{p})\to y \text{ as } t_k\to-\infty\}.$$

The following result completely characterizes the properties of the ω-limit set in the case of bounded solutions.

Lemma 4.8. *Suppose that $\gamma^+(\boldsymbol{p})$, $(\gamma^-(\boldsymbol{p}))$ is bounded. Then $\omega(\boldsymbol{p})$, $(\alpha(\boldsymbol{p}))$ is a closed, nonempty, connected, invariant set.*

Proof: The proof will only be given for $\omega(\boldsymbol{p})$, as that for $\alpha(\boldsymbol{p})$ is similar. Let $K\subset\mathbb{R}^n$ be a compact set such that $\gamma^+(p)\subset K$. By supposition, for each $\tau\ge0$

$$\overline{\{\phi_t(\boldsymbol{p}) \,:\, t\ge\tau\}}\subset K,$$

so that $\overline{\{\phi_t(\boldsymbol{p}) \,:\, t\ge\tau\}}$ is compact. In addition, $\overline{\{\phi_t(\boldsymbol{p}) \,:\, t\ge\tau\}}$ is connected for each $\tau\ge0$. Finally, for each $\tau_2>\tau_1$,

$$\overline{\{\phi_t(\boldsymbol{p}) \,:\, t\ge\tau_2\}}\subset\overline{\{\phi_t(\boldsymbol{p}) \,:\, t\ge\tau_1\}}.$$

Hence, $\omega(\boldsymbol{p})$ is the intersection of a nested family of compact connected sets, so by the Bolzano-Weierstrass theorem $\omega(\boldsymbol{p})$ is nonempty compact connected set.

Now let us show that $\omega(\boldsymbol{p})$ is invariant. Let $\boldsymbol{y}\in\omega(\boldsymbol{p})$ be given. By Remark 4.7 there exists an increasing sequence $\{t_n\}$ with $t_n\to\infty$ such that $\phi_{t_n}(\boldsymbol{p})\to\boldsymbol{y}$ as $n\to\infty$. By Lemma 4.2

$$\phi_{t_n+t}(\boldsymbol{p}) = \phi_t[\phi_{t_n}(\boldsymbol{p})],$$

so by continuity one gets that

$$\lim_{n\to\infty}\phi_{t_n+t}(\boldsymbol{p}) = \phi_t(\boldsymbol{y}).$$

Since $t_n+t\to\infty$ as $n\to\infty$ for any fixed $t\in\mathbb{R}$, one then has that

$$\lim_{n\to\infty}\phi_{t_n+t}(\boldsymbol{p})\in\omega(\boldsymbol{p});$$

hence, $\phi_t(\boldsymbol{y})\in\omega(\boldsymbol{p})$. $\square$

Critical points and periodic orbits correspond to invariant sets. What other type of orbits are to be found in $\omega(\boldsymbol{p})$? Two examples are:

Definition 4.9. Consider equation (4.1), where $\boldsymbol{f}(\boldsymbol{p}_0)=\boldsymbol{0}$ and $\boldsymbol{f}(\boldsymbol{p}_\pm)=\boldsymbol{0}$ for $\boldsymbol{p}_-\ne\boldsymbol{p}_+$. A homoclinic orbit satisfies

$$\lim_{t\to\pm\infty}\phi_t(\boldsymbol{x}_0)=\boldsymbol{p}_0.$$

A heteroclinic orbit satisfies

$$\lim_{t\to\pm\infty}\phi_t(\boldsymbol{x}_0)=\boldsymbol{p}_\pm.$$

4.2. Lyapunov functions

The goal is to determine the stability of the critical point through the use of generalized energy functions.

Definition 4.10. The C^1 function $V : \mathbb{R}^n \mapsto \mathbb{R}$ is positive definite if $V(\boldsymbol{0}) = 0$ and $V(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq \boldsymbol{0}$. The function is negative definite if $-V(\boldsymbol{x})$ is positive definite.

If $V(\boldsymbol{x})$ is positive definite, and if V is C^3, then one has $\nabla V(\boldsymbol{x}) = \boldsymbol{0}$ with $\sigma(\mathrm{D}^2 V(\boldsymbol{0})) = \sigma^{\mathrm{u}}(\mathrm{D}^2 V(\boldsymbol{0}))$. Furthermore, the level set $V(\boldsymbol{x}) = \epsilon$ has a "nice" component surrounding $\boldsymbol{x} = \boldsymbol{0}$ for $\epsilon > 0$ sufficiently small. For example, consider

$$V(\boldsymbol{x}) = \frac{1}{2} x_2^2 + \int_0^{x_1} g(s)\,\mathrm{d}s, \tag{4.2}$$

where $g(0) = 0$ and $yg(y) > 0$ for all $y \neq 0$. Since $yg(y) > 0$ implies that $\int_0^{x_1} g(s)\,\mathrm{d}s > 0$, this choice of V is a positive definite function.

Along trajectories one has that

$$\begin{aligned} \dot{V} &= \nabla V(\boldsymbol{x}) \cdot \dot{\boldsymbol{x}} \\ &= \nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \\ &= |\nabla V(\boldsymbol{x})|\,|\boldsymbol{f}(\boldsymbol{x})| \cos\theta. \end{aligned}$$

For example, for the planar system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -g(x_1),$$

the function given in equation (4.2) satisfies $\dot{V} = 0$, i.e., it is constant along trajectories. Since $\nabla V(\boldsymbol{x})$ is the outward pointing normal to the level set $V(\boldsymbol{x}) = \epsilon$, a minimal condition for the set bounded by $V(\boldsymbol{x}) = \epsilon$ to be invariant is that $\nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \leq 0$, as under this constraint the vector field is either parallel or points into the set. If $\dot{V} > 0$, then the vector field points out of the set. This observation leads to the following result:

Theorem 4.11 (Lyapunov's Stability Theorems)**.** *Suppose that $V : \mathbb{R}^n \mapsto \mathbb{R}$ is positive definite. Consider equation (4.1). If*

(a) $\nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \leq 0$, then $\boldsymbol{x} = \boldsymbol{0}$ is stable

(b) $\nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(x) < 0$, then $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable

(c) $\nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) > 0$, then $\boldsymbol{x} = \boldsymbol{0}$ is unstable.

Proof: (a) For each $r > 0$ set $B(r) := \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}| < r\}$. Let Ω be the region containing the origin such that $V(\boldsymbol{x})$ is positive definite on Ω with $\nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \leq 0$. Since $V(\boldsymbol{x})$ is positive definite, there is an $r_0 > 0$ such that $B(r_0) \subset \Omega$. Let $\boldsymbol{x}_0 \in B(r_0)$ be given, and let the solution emanating from $\boldsymbol{x}_0$ be denoted by $\phi(t)$. By Theorem 1.16 and Theorem 1.20 there is a $0 < \beta(\boldsymbol{x}_0) \leq +\infty$ such that $\beta(\boldsymbol{x}_0)$ is maximal and $\phi(t)$ exists for all $t \in [0, \beta(\boldsymbol{x}_0))$. By hypothesis and the Fundamental Theorem of Calculus,

$$V(\phi(t)) - V(\boldsymbol{x}_0) = \int_0^t \frac{\mathrm{d}V}{\mathrm{d}s}\,\mathrm{d}s = \int_0^t \nabla V(\phi(s)) \cdot \boldsymbol{f}(\phi(s))\,\mathrm{d}s \leq 0;$$

hence, $V(\phi(t)) \leq V(\boldsymbol{x}_0)$ for all $t \in [0, \beta(x_0))$. As a consequence of Theorem 1.16, $\phi(t) \neq 0$ for all t. Since $V(\boldsymbol{x})$ is positive definite, one can then conclude that $0 < V(\phi(t)) \leq V(\boldsymbol{x}_0)$.

Let $\epsilon > 0$ be given with $0 < \epsilon \leq r_0$, and set

$$S_\epsilon := \{\boldsymbol{x} \in \mathbb{R}^n : \epsilon \leq |\boldsymbol{x}| \leq r_0\}.$$

Since $V(\boldsymbol{x})$ is continuous and S_ϵ is closed, there exists a

$$0 < \mu := \min_{\boldsymbol{x} \in S_\epsilon} V(\boldsymbol{x}).$$

The left-hand inequality arises since $V(\boldsymbol{x})$ is positive definite. Since $V(\boldsymbol{0}) = 0$, there is a $0 < \delta < \mu$ such that if $|\boldsymbol{x}_0| < \delta$, then $V(\boldsymbol{x}_0) < \mu$. Thus, if $|\boldsymbol{x}_0| < \delta$, $0 < V(\phi(t)) < \mu$ for all $t \in [0, \beta(\boldsymbol{x}_0))$. By the definition of μ

this implies that $\phi(t) \notin S_\epsilon$, so that $|\phi(t)| < \epsilon$ for all $t \in [0, \beta(\boldsymbol{x}_0))$. Hence, $\beta(\boldsymbol{x}_0) = +\infty$, and the solution is stable.

(b) Since $\boldsymbol{x} = \boldsymbol{0}$ is stable, it must now be shown that $\lim_{t\to+\infty} \phi(t) = 0$. Since $V(\boldsymbol{x})$ is positive definite, it is enough to show that $\lim_{t\to+\infty} V(\phi(t)) = 0$. Suppose that there exists an $0 < \eta < r_0$ such that $V(\phi(t)) \geq \eta$ for all $t \geq 0$. Since $V(\boldsymbol{x})$ is continuous, there is a $\delta > 0$ such that if $|\boldsymbol{x}| < \delta$, then $V(\boldsymbol{x}) < \eta$. Since $V(\phi(t)) \geq \eta$, it must be true that $|\phi(t)| \geq \delta$ for all $t \geq 0$. Set

$$S_\delta := \{\boldsymbol{x} \in \mathbb{R}^n : \delta \leq |\boldsymbol{x}| \leq r_0\},$$

and consider $V^*(\boldsymbol{x}) := -\nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x})$ on S_δ. By hypothesis $V^*(\boldsymbol{x}) > 0$ is continuous on S_δ, so that there is a

$$0 < \mu := \min_{\boldsymbol{x}\in S_\delta} V^*(\boldsymbol{x}).$$

Since $\boldsymbol{0} \notin S_\delta$, and since $\phi(t) \in S_\delta$ for all $t \geq 0$, one has that

$$-\dot{V}(\phi(t)) = V^*(\phi(t)) \geq \mu.$$

From the Fundamental Theorem of Calculus one then gets that

$$V(\boldsymbol{x}_0) - V(\phi(t)) = \int_0^t V^*(\phi(s))\,\mathrm{d}s \geq \mu t;$$

hence, $V(\phi(t)) \leq V(\boldsymbol{x}_0) - \mu t$. For $t > V(\boldsymbol{x}_0)/\mu$ one then has that $V(\phi(t)) < 0$, which is a contradiction.

(c) By the Fundamental Theorem of Calculus one has that for each $t_2 > t_1 \geq 0$,

$$V(\phi(t_2)) - V(\phi(t_1)) = \int_{t_1}^{t_2} \nabla V(\phi(s)) \cdot \boldsymbol{f}(\phi(s))\,\mathrm{d}s > 0;$$

hence, $V(\phi(t))$ is a strictly increasing function. Let $T > 0$ be the first time that $|\phi(T)| = r_0$, and if no such T exists, set $T = +\infty$. As in the proof of (b), $|\phi(t)| \geq \delta$ for all $t \in [0, T]$; hence, on the set S_δ one has $\nabla V(\phi(t)) \cdot \boldsymbol{f}(\phi(t)) \geq \mu$ and consequently $V(\phi(t)) \geq V(\boldsymbol{x}_0) + \mu t$. Since $V(\boldsymbol{x}) \leq M$ for $|\boldsymbol{x}| \leq r_0$, this yields $T < +\infty$. □

Remark 4.12. The statement of part Theorem 4.11(c) can be weakened in the following manner [**4**, Exercise 1.38]. Suppose that $V(\boldsymbol{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ is C^1 and satisfies

(a) $V(\boldsymbol{0}) = 0$

(b) $\nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) > 0$

(c) $V(\boldsymbol{x})$ takes positive values in each sufficiently small neighborhood of $\boldsymbol{x} = \boldsymbol{0}$.

Then $\boldsymbol{x} = \boldsymbol{0}$ is unstable.

For an example, consider equation (4.1) under the assumptions that $\sigma(\boldsymbol{A}) \subset \mathbb{R}$, and that each $\lambda \in \sigma(\boldsymbol{A})$ is semi-simple. As previously discussed, the second assumption is generic. There then exists a nonsingular matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \Lambda$, where $\Lambda := \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Upon setting $\boldsymbol{x} = \boldsymbol{P}\boldsymbol{y}$, equation (4.1) becomes the system

$$\dot{\boldsymbol{y}} = \Lambda \boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y}), \quad \boldsymbol{g}(y) := \boldsymbol{P}^{-1}\boldsymbol{r}(\boldsymbol{P}\boldsymbol{y}). \tag{4.3}$$

Note that $|\boldsymbol{g}(\boldsymbol{y})| = \mathcal{O}(|\boldsymbol{y}|^2)$ for $\boldsymbol{y}$ sufficiently close to the origin. Furthermore, since $\boldsymbol{P}$ is nonsingular, any stability statements made regarding equation (4.3) immediately apply to equation (4.1).

Now define the positive definite function

$$V(\boldsymbol{y}) := \frac{1}{2}\sum_{i=1}^n y_i^2.$$

One has that for equation (4.3),

$$\dot{V}(t) = \sum_{i=1}^n y_i \dot{y}_i = \sum_{i=1}^n \lambda_i y_i^2 + \sum_{i=1}^n y_i g_i(\boldsymbol{y}).$$

For each $\epsilon > 0$ there is a $\delta > 0$ such that if $|\boldsymbol{y}| < \delta$, then $|g_i(\boldsymbol{y})| < \epsilon|\boldsymbol{y}|$. This follows from the fact that $|\boldsymbol{g}(\boldsymbol{y})| = \mathcal{O}(|\boldsymbol{y}|^2)$. Suppose that $\sigma(\boldsymbol{A}) = \sigma^{\mathrm{s}}(\boldsymbol{A})$; hence, there exists a $\mu \in \mathbb{R}^+$ such that $\lambda_i \le -\mu < 0$ for all i. If $\epsilon < \mu/2$, then one has that

$$\dot{V}(t) \le \sum_{i=1}^{n}(\lambda_i + \epsilon)y_i^2 < -\frac{\mu}{2}|\boldsymbol{y}|^2 < 0,$$

so by Theorem 4.11 $\boldsymbol{y} = \boldsymbol{0}$ is asymptotically stable. Similarly, it can be shown that if $\sigma(\boldsymbol{A}) = \sigma^{\mathrm{u}}(\boldsymbol{A})$, then $\boldsymbol{y} = \boldsymbol{0}$ is unstable, as in this case $\dot{V}(t) > 0$ for $|\boldsymbol{y}| < \epsilon$.

Remark 4.13. It is an exercise for the student to show that a saddle is unstable. The proof requires an application of Remark 4.12.

The result of Theorem 4.11 is local in the sense that a definitive statement can be made only in a sufficiently small neighborhood of a critical point. The next result gives one possible way to make the result more global; furthermore, it precisely locates the ω-limit sets.

Theorem 4.14 (Invariance principle)**.** *Consider equation (4.1). Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be positive definite, and for each $k \in \mathbb{R}^+$ set*

$$U_k := \{\boldsymbol{x} \in \mathbb{R}^n \,:\, V(\boldsymbol{x}) < k\}.$$

Suppose that $\nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \le 0$ on U_k. Set

$$S := \{\boldsymbol{x} \in U_k \,:\, \nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) = 0\}.$$

For each $\boldsymbol{x}_0 \in U_k$ one has that $\omega(\boldsymbol{x}_0) \subset S$. In particular, if $\{\boldsymbol{0}\} \subset S$ is the largest invariant set in U_k, then $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable.

Proof: The proof requires the material presented in Section 4.1. Since $\nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \le 0$ on U_k, one has that the set U_k is invariant under the flow; furthermore, $V(\phi_t(\boldsymbol{x}_0)) \le V(\phi_s(\boldsymbol{x}_0)) \le V(\boldsymbol{x}_0)$ for all $s < t \in \mathbb{R}^+$. For a given $\boldsymbol{x}_0 \in U_k$, let $\boldsymbol{p} \in \omega(\boldsymbol{x}_0)$. The existence of such a point is guaranteed by Lemma 4.8. By Definition 4.6 one further has that $V(\boldsymbol{p}) \le V(\phi_t(\boldsymbol{x}_0))$ for any $t \in \mathbb{R}^+$. Let $\{t_n\} \subset \mathbb{R}^+$ be a monotone increasing sequence with $t_n \to +\infty$ as $n \to +\infty$ such that $\phi_{t_n}(\boldsymbol{x}_0) \to \boldsymbol{p}$ as $n \to +\infty$. By continuity one has that $V(\phi_{t_n}(\boldsymbol{x}_0)) \to V(\boldsymbol{p})$ as $n \to +\infty$. The continuous dependence of solutions on initial data (see Theorem 1.17) implies that for $n \in \mathbb{N}$ sufficiently large and $t \in \mathbb{R}$ sufficiently small one has that $|\phi_{t+t_n}(\boldsymbol{x}_0) - \phi_t(\boldsymbol{p})|$ is small. Consequently, by continuity one has $|V(\phi_{t+t_n}(\boldsymbol{x}_0)) - V(\phi_t(\boldsymbol{p}))|$ is small for $n \in \mathbb{N}$ sufficiently large and $t \in \mathbb{R}$ sufficiently small. Suppose that $\boldsymbol{p} \notin S$, so that $\dot{V}(\boldsymbol{p}) < 0$. One then has that

$$V(\phi_t(\boldsymbol{p})) < V(\boldsymbol{p}) < V(\phi_{-t}(\boldsymbol{p})), \quad 0 < t \ll 1,$$

so by continuity one has that for $n \in \mathbb{N}$ sufficiently large,

$$V(\phi_{t_n+t}(\boldsymbol{x}_0)) < V(\phi_{t_n}(\boldsymbol{x}_0)) < V(\phi_{t_n-t}(\boldsymbol{x}_0)).$$

Continuity then yields that $V(\phi_{t_n+t}(\boldsymbol{x}_0)) < V(\boldsymbol{p})$ for $n \in \mathbb{N}$ sufficiently large and $0 < t \ll 1$ sufficiently small. This is a contradiction; hence, one must have $\boldsymbol{p} \in S$.

The stability of $\boldsymbol{x} = \boldsymbol{0}$ follows immediately from Theorem 4.11. Since $\omega(\boldsymbol{x}_0) \subset S$, one has that $\mathrm{dist}(\phi_t(\boldsymbol{x}_0), S) \to 0$ as $t \to +\infty$. If $\{\boldsymbol{0}\} \subset S$ is the largest invariant set in U_k, then one gets that $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable. □

For an example, consider van der Pol's equation,

$$\ddot{x} + 2\mu(1 - x^2)\dot{x} + x = 0, \quad \mu > 0.$$

In Lienard form it is written as

$$\dot{x}_1 = -2\mu x_1(1 - x_1^2/3) + x_2, \quad \dot{x}_2 = -x_1.$$

Consider the positive definite function $V(\boldsymbol{x}) = (x_1^2 + x_2^2)/2$. One has that

$$\nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) = -2\mu x_1^2(1 - x_1^2/3).$$

If one sets

$$U_{3/2} = \{\boldsymbol{x} \in \mathbb{R}^2 \,:\, V(\boldsymbol{x}) < 3/2\},$$

then $\nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) \leq 0$ for $\boldsymbol{x} \in U_{3/2}$. Set

$$S = \{\boldsymbol{x} \in U_{3/2} \,:\, x_1 = 0\}.$$

Now, $\dot{x}_1 = x_2 \neq 0$ except at $(0,0)$. Hence, the origin is the largest invariant set in S, so by Theorem 4.14 $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable.

4.2.1. Example: Hamiltonian systems

Hamilton's equations of motion are given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, n, \tag{4.4}$$

where $H = H(\boldsymbol{p}, \boldsymbol{q}) \in C^2(\mathbb{R}^{2n})$. Note that upon setting $\boldsymbol{x} := (\boldsymbol{q}, \boldsymbol{p})^{\mathrm{T}}$ equation (4.4) can be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{J} \nabla H(\boldsymbol{x}), \quad \boldsymbol{J} := \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{4.5}$$

For example, if one considers

$$\ddot{x} + f(x) = 0 \tag{4.6}$$

then by setting $(q, p) := (x, \dot{x})$ one has the Hamiltonian

$$H(p, q) = \frac{1}{2} p^2 + F(q), \quad F(q) := \int_0^q f(s)\,\mathrm{d}s. \tag{4.7}$$

In equation (4.7) one has the physical interpretation that $p^2/2$ is the kinetic energy and $F(q)$ is the potential energy. One has that

$$\dot{H} = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} \dot{q}_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \dot{p}_i = 0;$$

hence, one can use the Hamiltonian as a Lyapunov function. Without loss of generality one can assume that $H(\boldsymbol{0}) = 0$. If $\nabla H(\boldsymbol{0}) = \boldsymbol{0}$, i.e., if $\boldsymbol{x} = \boldsymbol{0}$ is a critical point for equation (4.5), and if H is positive definite, i.e., $\sigma(\mathrm{D}^2 H(\boldsymbol{0})) = \sigma^{\mathrm{u}}(\mathrm{D}^2 H(\boldsymbol{0}))$, then by Theorem 4.11 the origin is stable. The conclusion still follows if H is negative definite, i.e., $\sigma(\mathrm{D}^2 H(\boldsymbol{0})) = \sigma^{\mathrm{s}}(\mathrm{D}^2 H(\boldsymbol{0}))$, if instead of taking H as the Lyapunov function one takes $-H$.

Now assume that

$$H(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \phi(\boldsymbol{q}),$$

where $\phi(\boldsymbol{0}) = \boldsymbol{0}$ and $\nabla \phi(\boldsymbol{0}) = \boldsymbol{0}$. Upon a change of coordinates one can write

$$\phi(\boldsymbol{q}) = \frac{1}{2} \sum_{i=1}^{n} a_i q_i^2 + \mathcal{O}(|\boldsymbol{q}|^3).$$

In this new coordinate system the Hamiltonian equations are

$$\dot{q}_i = p_i, \quad \dot{p}_i = -a_i q_i + \mathcal{O}(|\boldsymbol{q}|^2).$$

The linearization about the critical point yields the eigenvalues $\lambda = \pm\sqrt{-a_i}$. Note that these eigenvalues are generally semi-simple. If ϕ is positive definite, i.e., if $a_i \in \mathbb{R}^+$ for each i, then it is easy to show that H is positive definite; hence, the origin is stable. If $a_i \in \mathbb{R}^-$ for some (but not all) i, i.e., if ϕ is not positive definite at $\boldsymbol{q} = \boldsymbol{0}$, then the origin is a saddle point, and by Remark 4.13 is consequently unstable.

For example, again consider equation (4.6), with the associated Hamiltonian given in equation (4.7). Assume that $F'(0) = f(0) = 0$, and note that $F''(0) = f'(0)$. From the above discussion one has that if $f'(0) < 0$, then the origin is unstable, whereas if $f'(0) > 0$ the origin is stable. Thus, minima of F correspond to stable critical points, while maxima correspond to unstable critical points. This corresponds to the physical intuition that minimum points of the potential energy are stable, while maximum points are unstable.

5. Periodic Solutions

Again consider the autonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \tag{5.1}$$

where $\boldsymbol{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is smooth. Recall that from Lemma 4.8 it is known that for bounded trajectories the ω-limit set is compact, connected, and invariant. In Section 4 some conditions were given which guaranteed that this set was the critical point $\boldsymbol{x} = \boldsymbol{0}$. In this section we will be concerned with the existence and nonexistence of periodic solutions. Most of the results given herein will be applicable only in the case that $n = 2$, as the topology of the plane allows one to make more definitive statements regarding the ω-limit set. In fact, unless otherwise stated it will henceforth be assumed that $n = 2$.

5.1. *Nonexistence: Bendixson's criterion*

In this subsection we will give a criteria which guarantees that no solutions exist to equation (5.1). Before doing so, however, one needs to be re-acquainted with Green's Theorem and the Divergence Theorem. This in turn requires the following characterization of closed continuous curves $\gamma \subset \mathbb{R}^2$.

Theorem 5.1 (Jordan Curve Theorem)**.** *A simple closed continuous curve $\gamma \subset \mathbb{R}^2$ divides the plane into two connected components. One is bounded, and is called the interior of γ, and the other is unbounded and called the exterior of the γ. Each component has γ as its boundary.*

Definition 5.2. If $\gamma \subset \mathbb{R}^2$ satisfies the Jordan Curve Theorem, set $\mathrm{int}(\gamma)$ to be the interior of γ, and $\mathrm{ext}(\gamma)$ to be the exterior of γ.

Theorem 5.3. *Let $\boldsymbol{f} = (f_1, f_2) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be smooth. Let $\gamma \subset \mathbb{R}^2$ be a Jordan curve bounding a domain D. One has the following results:*

(a) *Green's theorem:* $\displaystyle\oint_\gamma \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{R} = \iint_D (\partial_{x_1} f_2 - \partial_{x_2} f_1)\, \mathrm{d}A$

(b) *Divergence theorem:* $\displaystyle\oint_\gamma f_1 \mathrm{d}x_2 - f_2 \mathrm{d}x_1 = \iint_D \nabla \cdot \boldsymbol{f}\, \mathrm{d}A,$

where $\nabla \cdot \boldsymbol{f} := \partial_{x_1} f_1 + \partial_{x_2} f_2$.

If in Theorem 5.3 one thinks of γ as representing a invariant Jordan curve, e.g., a periodic orbit, for equation (5.1), then one can interpret Green's Theorem and the Divergence Theorem as giving necessary conditions on the vector field for the existence of γ. In particular:

Theorem 5.4 (Bendixson's criterion)**.** *Consider equation (5.1) when $n = 2$. Let $\Omega \subset \mathbb{R}^2$ be a simply connected region. If $\nabla \cdot \boldsymbol{f}(\boldsymbol{x}) \neq 0$ for all $\boldsymbol{x} \in \Omega$, then the system has no invariant Jordan curves contained in Ω.*

Proof: Suppose that there is an invariant Jordan curve $\gamma \subset \Omega$. Parameterize the curve so that it is traversed once in the counterclockwise direction for $0 \leq t \leq 1$. Set $D := \mathrm{int}(\gamma)$. Since the curve is invariant one has that the vector field $\boldsymbol{f}$ is tangent at all points; consequently, one can say without loss of generality that $\dot{x}_i = f_i(\boldsymbol{x})$ along the curve. One then has that

$$\oint_\gamma f_1 \mathrm{d}x_2 - f_2 \mathrm{d}x_1 = \int_0^1 (f_1 \frac{\mathrm{d}x_2}{\mathrm{d}t} - f_2 \frac{\mathrm{d}x_1}{\mathrm{d}t})\, \mathrm{d}t = 0.$$

By Theorem 5.3(b) this then implies that

$$\iint_D \nabla \cdot \boldsymbol{f}\, \mathrm{d}A = 0,$$

which contradicts the assumption that $\nabla \cdot \boldsymbol{f}$ never changes sign. Hence, γ does not exist. $\square$

For the first example, recall that in Section 4.2 van der Pol's equation, in Lienard form, was given by

$$\dot{x}_1 = -2\mu x_1(1 - x_1^2/3) + x_2, \quad \dot{x}_2 = -x_1.$$

Further recall that if $\boldsymbol{x}(t) \in U_{\mathrm{s}} := \{\boldsymbol{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 < 3\}$ for any value of t, then $\boldsymbol{x}(t) \in U_{\mathrm{s}}$ for all $t \in \mathbb{R}$ with $\boldsymbol{x}(t) \to \boldsymbol{0}$ as $t \to \infty$. Now, $\nabla \cdot \boldsymbol{f} = -2\mu(1 - x_1^2)$. Upon setting

$$U_+^1 := \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 > 1\}, \quad U_-^1 := \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 < -1\},$$

an application of Theorem 5.4 then yields that any periodic solution γ must satisfy $\gamma \subset \mathbb{R}^2 \backslash U_{\mathrm{s}}$ with $\gamma \not\subset U_\pm^1$ and either (or both) $\gamma \cap U_\pm^1 \neq \varnothing$.

For the second example, consider

$$\ddot{x} + p(x)\dot{x} + q(x) = 0.$$

Suppose that $p(x) > 0$ (damping). For the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -q(x_1) - p(x_1)x_2$$

one has that $\nabla \cdot \boldsymbol{f} = -p(x_1) < 0$. Hence, by Theorem 5.4 no periodic solution exists. In fact, if $q(0) = 0$ and $xq(x) > 0$ for all $x \neq 0$, then by using the appropriate Lyapunov function one has that for any x_0, $x(t) \to 0$ as $t \to \infty$.

5.2. Existence: Poincaré-Bendixson Theorem

Now that one has criteria for which no periodic solutions exist, it is time to develop conditions under which one can guarantee the existence of such solutions. The notation and ideas presented in Section 4.1 will be used extensively here.

Again consider equation (5.1) in the case that $n = 2$. Let $\gamma := \{\phi_t(p) : 0 \le t \le T\}$ be a periodic orbit with minimal period T. Let $\boldsymbol{v} \in \mathbb{R}^2$ be chosen so that $\boldsymbol{v} \cdot \boldsymbol{f}(\boldsymbol{p}) = 0$. The vector $\boldsymbol{v}$ is said to be transversal to γ at the point $\boldsymbol{p}$. For $\epsilon > 0$ set

$$L_\epsilon := \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{x} = \boldsymbol{p} + a\boldsymbol{v},\ |a| \le \epsilon\}.$$

L_ϵ is said to be a transversal section to γ at p. Since $\boldsymbol{f}(\boldsymbol{p}) \neq \boldsymbol{0}$, $\epsilon > 0$ can be chosen sufficiently small so that $L_\epsilon \cap \gamma = \{\boldsymbol{p}\}$, and that all orbits crossing L_ϵ do so in the same direction.

Lemma 5.5. *There is a $\delta > 0$ such that if $\boldsymbol{x}_0 \in L_\delta$, then there is a continuous $T(\boldsymbol{x}_0) > 0$ with $\lim_{\boldsymbol{x}_0 \to \boldsymbol{p}} T(\boldsymbol{x}_0) = T$ such that*

$$\phi_{T(\boldsymbol{x}_0)}(\boldsymbol{x}_0) \in L_\epsilon.$$

Proof: Since $\boldsymbol{f}$ is smooth, the flow $\phi_t(\boldsymbol{x})$ depends smoothly on $\boldsymbol{x}$. Applying the Implicit Function Theorem to $G(t, \boldsymbol{x}) := \boldsymbol{v} \cdot \boldsymbol{f}(\phi_t(\boldsymbol{x}))$ (note that $G(T, \boldsymbol{p}) = 0$) yields the result. □

Remark 5.6. If $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, the transversal section is defined by

$$\Sigma := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = \boldsymbol{p} + \sum_{i=1}^{n-1} a_i \boldsymbol{v}_i,\ |a_i| \le \epsilon\},$$

where $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n-1}\}$ is a linearly independent set which satisfies $\boldsymbol{v}_i \cdot \boldsymbol{f}(\boldsymbol{p}) = 0$ for all i.

Lemma 5.5 allows one to define a smooth map near a periodic orbit.

Definition 5.7. The return time map is given by $T(\boldsymbol{x})$, and the Poincaré map is given by $\Pi(\boldsymbol{x}) = \phi_{T(\boldsymbol{x})}(\boldsymbol{x})$.

Lemma 5.5 allows one to understand the dynamics near a periodic orbit via a study of a map, versus the study of the full flow. The advantage to this approach is that the dimensionality is reduced by one. However, even in the case that $n = 2$ this does not necessarily imply that the problem is easy (e.g., see [7, Chapter 3]). However, one recovers periodic orbits quite easily as fixed points of Π.

Lemma 5.8. *Let $\boldsymbol{x}_0 \in \Sigma$, where Σ is defined in Remark 5.6. $\Pi(\boldsymbol{x}_0) = \boldsymbol{x}_0$ if and only if $\phi_t(\boldsymbol{x}_0)$ is a periodic orbit with minimal period $T(\boldsymbol{x}_0)$.*

Proof: By definition, $\Pi(\boldsymbol{x}_0) = \boldsymbol{x}_0$ if and only if $\phi_{T(\boldsymbol{x}_0)}(\boldsymbol{x}_0) = \boldsymbol{x}_0$ for some $T(\boldsymbol{x}_0) > 0$. The result now follows from Theorem 1.16. $\square$

Consider equation (5.1) when $n = 2$. Suppose that there is a section $L \subset \mathbb{R}^2$ such that for the orbit $\phi_t(\boldsymbol{p})$ one can define a Poincaré map $\Pi : L \mapsto L$. Without loss of generality it can be assumed that L is a subset of the x_1-axis, so that the map can be represented by $(x_1, 0) \mapsto (\Pi(x_1), 0)$. By Theorem 1.16 and smoothness one has that the map is a diffeomorphism. For a given point $(x_1, 0) \in L$, one has three possibilities for the map: (a) $\Pi(x_1) = x_1$, (b) $\Pi(x_1) > x_1$, or (c) $\Pi(x_1) < x_1$. Case (a) implies that $(x_1, 0)$ is contained in a periodic orbit. The uniqueness of solutions implies in cases (b) and (c) that the sequence $\{\Pi^j(x_1)\}$ will either be monotone increasing (case (b)) or decreasing (case (c)) for all $j \in \mathbb{N}$. If one assumes that $\gamma^+(\boldsymbol{p})$ is bounded, then one has that the sequence is bounded, and hence has a limit point. The following theorem allows one to characterize the orbits associated with these limit points.

Theorem 5.9 (Poincaré-Bendixson Theorem)**.** *Consider equation (5.1) under the assumptions that $n = 2$ and that there exist only a finite number of critical points. Suppose that for some $\boldsymbol{p} \in \mathbb{R}^2$, $\gamma^+(\boldsymbol{p})$ is bounded. Then one of the following holds:*

(a) $\omega(\boldsymbol{p})$ is a critical point

(b) $\omega(\boldsymbol{p})$ is a periodic orbit

(c) $\omega(\boldsymbol{p})$ is the union of finitely many critical points and perhaps a countably infinite set of connecting orbits

In cases (b) and (c) $\omega(\boldsymbol{p})$ satisfies the Jordan Curve Theorem.

Proof: E.g., see [**17**, Chapter 4.3]. The basic idea is to carefully study the properties of the relevant Poincaré map. $\square$

Remark 5.10. One has that:

(a) if $\gamma^-(\boldsymbol{p})$ is bounded, then there is a similar result for $\alpha(\boldsymbol{p})$

(b) case (c) allows the existence of heteroclinic and homoclinic orbits

An easy consequence of Theorem 5.9, which is especially useful in applications, is the following:

Corollary 5.11. *If there is a positively invariant region Ω which contains no critical points, then Ω contains at least one periodic orbit.*

If $\gamma \subset \mathbb{R}^2$ is a periodic orbit, one can characterize its stability in the following manner.

Definition 5.12. A periodic orbit $\gamma \subset \mathbb{R}^2$ is a limit cycle if there is a $\boldsymbol{p}_1 \in \mathrm{int}(\gamma)$ and $\boldsymbol{p}_2 \in \mathrm{ext}(\gamma)$ such that either $\omega(\boldsymbol{p}_1) = \omega(\boldsymbol{p}_2) = \gamma$ or $\alpha(\boldsymbol{p}_1) = \alpha(\boldsymbol{p}_2) = \gamma$.

5.2.1. Examples

Example (**I**)**.** Consider

$$\dot{x}_1 = \beta x_1 - x_2 + (3x_1^2 + 2x_2^2)x_1, \quad \dot{x}_2 = x_1 + \beta x_2 + (3x_1^2 + 2x_2^2)x_2.$$

In polar coordinates $(x_1 := r\cos\theta, x_2 := r\sin\theta)$ the system is

$$\dot{r} = r(\beta + (2 + \cos^2\theta)r^2), \quad \dot{\theta} = 1.$$

If $\beta > 0$, then $\dot{r} > 0$, so that all solutions are unbounded as $t \to +\infty$; hence, there exist no periodic orbits. As $t \to -\infty$ all trajectories approach the origin.

Suppose that $\beta < 0$. For $0 < \epsilon \ll 1$ consider the annulus

$$D_\epsilon := \{(r,\theta) \,:\, -\frac{\beta}{3-\epsilon} < r^2 < -\frac{\beta}{2+\epsilon}\}.$$

Since $r(\beta+2r^2) < \dot{r} < r(\beta+3r^2)$, D_ϵ is negatively invariant, i.e., the vector field points out on the boundary of D_ϵ. Thus, if $\boldsymbol{p} \in D_\epsilon$, $\gamma^-(\boldsymbol{p})$ satisfies the hypotheses of the Poincaré-Bendixson theorem. Since D_ϵ contains no critical points, $\alpha(\boldsymbol{p})$ is a periodic orbit.

Unfortunately, the Poincaré-Bendixson theorem says nothing about the number of periodic orbits in D_ϵ. However, since D_ϵ is an annulus (which is *not* simply connected), by appropriately modifying the proof to the Bendixson criterion (see [**15**, p. 262]) one cay say the following.

Lemma 5.13. *Consider $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, where $\boldsymbol{f} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is smooth. Suppose that there is an annular region $\Omega \subset \mathbb{R}^2$ such that $\nabla \cdot \boldsymbol{f}(\boldsymbol{x}) \neq 0$ for all $\boldsymbol{x} \in \Omega$. Then there exists at most one periodic orbit $\gamma \subset \Omega$.*

Example (**I** (cont.))**.** A routine calculation shows that

$$\nabla \cdot \boldsymbol{f}(\boldsymbol{x}) = 2(\beta + 2(2+\cos^2\theta)r^2),$$

so that $\nabla \cdot \boldsymbol{f}(\boldsymbol{x}) > 0$ for $r^2 > -\beta/4$. Thus, $\nabla \cdot \boldsymbol{f}(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in D_\epsilon$, so that by Lemma 5.13 there is only one periodic solution contained in D_ϵ. Also, $\dot{r} < 0$ for $r^2 < -\beta/3$ and $\dot{r} > 0$ for $r^2 > -\beta/2$ implies that neither of these regions contains a periodic orbit. Hence, the periodic orbit contained in D_ϵ is unique.

Remark 5.14. In general, an invariant region may contain more than one periodic orbit. Consider

$$\dot{r} = -r(r-1)(r-2), \quad \dot{\theta} = 1.$$

The annulus

$$D := \{(r,\theta) \,:\, \frac{1}{2} < r^2 < \frac{5}{2}\}$$

is positively invariant, and contains the two periodic orbits $r = 1, 2$.

Example (**II**)**.** Recall that Van der Pol's equation, in Lienard form, is given by

$$\dot{x}_1 = -2\mu x_1(1 - x_1^2/3) + x_2, \quad \dot{x}_2 = -x_1.$$

Suppose that $\mu < 0$. It can be shown that the set $S_{\text{neg}} := \{x \in \mathbb{R}^2 \,:\, x_1^2 + x^2 < 3\}$ is negatively invariant; furthermore, if $x_0 \in S_{\text{neg}}$, then $x(t) \to 0$ as $t \to -\infty$. Furthermore, it has been seen that any periodic solution must intersect either (or both) of $x_1 = \pm 1$.

Let us now show that such a periodic solution exists in a particular limit. Set $\epsilon := 1/|\mu|$, $\hat{x}_2 := \epsilon x_2$, and rewrite the equations (upon removing the hat) as

$$\epsilon\dot{x}_1 = 2x_1(1 - x_1^2/3) + x_2, \quad \dot{x}_2 = -\epsilon x_1.$$

This is a singular system, for the vector field is not smooth as $\epsilon \to 0^+$. Set $s := t/\epsilon$, so that the equations now become ($' := \mathrm{d}/\mathrm{d}s$)

$$x_1' = 2x_1(1 - x_1^2/3) + x_2, \quad x_2' = -\epsilon^2 x_1.$$

Let us first study this new system in the case $\epsilon = 0$. The lines $x_2 = C$ are invariant, and on these lines the ODE is given by

$$x_1' = 2x_1(1 - x_1^2/3) + x_2.$$

Thus, one can construct an invariant Jordan curve, say γ, composed of critical points and heteroclinic orbits. It can be shown that given $\delta > 0$ there is an $\epsilon_0 > 0$ and an open set U lying within a distance δ of γ such that U is positively invariant for $0 < \epsilon < \epsilon_0$. This set U will contain no critical points, so by the Poincaré-Bendixson theorem there will exist a periodic orbit in U. The proof is by picture, and uses the fact that $x_2' = -\epsilon^2 x_1$ for $\epsilon > 0$. The resulting periodic solution is an example of a relaxation-oscillation (a periodic solution operating on different time scales). See [**17**, Chapter 12.3] for further details.

***Example* (III).** Let γ_1 and γ_2 be two invariant Jordan curves, and suppose that $\gamma_1 \subset \mathrm{int}(\gamma_2)$. Set $D := \mathrm{int}(\gamma_2) \cap \mathrm{ext}(\gamma_1)$. Suppose that there exist no critical points or periodic orbits in D. Let $\boldsymbol{x}_0 \in D$ be given. Since D is bounded and invariant, $\omega(\boldsymbol{x}_0) \subset \overline{D}$. Since D contains no critical points or periodic orbits, by the Poincaré-Bendixson theorem one has that either $\omega(\boldsymbol{x}_0) = \gamma_1$ or $\omega(\boldsymbol{x}_0) = \gamma_2$. Without loss of generality, suppose that $\omega(\boldsymbol{x}_0) = \gamma_1$. Let us now show that for $\boldsymbol{p} \in D$, $\omega(\boldsymbol{p}) = \gamma_1$.

Suppose that there is an $\boldsymbol{x}_1 \in D$ such that $\omega(\boldsymbol{x}_1) = \gamma_2$. Let ℓ be the line containing $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ which transversely intersects γ_1 and γ_2. Pick a point $\boldsymbol{x}_2 \in \ell$ which is between $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$. By the uniqueness of solutions, $\gamma^-(\boldsymbol{x}_2)$ is trapped between $\gamma^+(\boldsymbol{x}_0)$ and $\gamma^+(\boldsymbol{x}_1)$, which implies that $\gamma^-(\boldsymbol{x}_2)$ is uniformly bounded away from both γ_1 and γ_2. By the Poincaré-Bendixson theorem $\alpha(\boldsymbol{x}_2)$ is either a periodic orbit or contains critical points. Since $\alpha(\boldsymbol{x}_2) \subset D$ and $\alpha(\boldsymbol{x}_2) \cap (\gamma_1 \cup \gamma_2) = \varnothing$, this implies that D itself contains either a periodic orbit or critical points. This is a contradiction; thus, $\omega(\boldsymbol{x}_1) = \gamma_1$.

5.3. *Index theory*

Given a periodic orbit $\gamma \subset \mathbb{R}^2$, it is natural to inquire as to what types of orbits reside in $\mathrm{int}(\gamma)$. In particular, does $\mathrm{int}(\gamma)$ necessarily contain critical points? If so, is the nature of the flow near the critical point necessarily proscribed? An application of the following theorem yields an answer to the first question.

Theorem 5.15 (Brouwer's fixed point theorem)**.** *Let $U \subset \mathbb{R}^n$ be homeomorphic to a closed ball in $\mathbb{R}^n$. Let $\boldsymbol{g} : U \mapsto \mathbb{R}^n$ be continuous and satisfy $\boldsymbol{g}(\partial U) \subset U$. Then $\boldsymbol{g}$ has at least one fixed point in U, i.e., there is at least one $\boldsymbol{x}^* \in U$ such that $\boldsymbol{g}(\boldsymbol{x}^*) = \boldsymbol{x}^*$.*

Theorem 5.16. *Let $\gamma \subset \mathbb{R}^2$ be an invariant Jordan curve. Then $\mathrm{int}(\gamma)$ contains at least one critical point.*

Proof: If $\gamma \subset \mathbb{R}^2$ is a Jordan curve, then $\mathrm{int}(\gamma)$ is homeomorphic to a closed ball in $\mathbb{R}^2$. Since the flow is continuous and satisfies $\phi_t(\mathrm{int}(\gamma)) = \mathrm{int}(\gamma)$, one can attempt to apply Theorem 5.15 to deduce the existence of an equilibrium point.

For a given $\boldsymbol{p} \in \mathrm{int}(\gamma)$ one has that $\phi_t(\boldsymbol{p}) \in \mathrm{int}(\gamma)$ for all $t \geq 0$; in particular, this implies that for a given $t_1 > 0$ one has that $\phi_{t_1} : \mathrm{int}(\gamma) \mapsto \mathrm{int}(\gamma)$. Thus, by Theorem 5.15 there is a $\boldsymbol{p}_1 \in \mathrm{int}(\gamma)$ such that $\phi_{t_1}(\boldsymbol{p}_1) = \boldsymbol{p}_1$. Choose a decreasing sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = 0$, and get the corresponding sequence $\{\boldsymbol{p}_n\}$ with $\phi_{t_n}(\boldsymbol{p}_n) = \boldsymbol{p}_n$.

Without loss of generality, suppose that $\lim_{n\to\infty} \boldsymbol{p}_n = \boldsymbol{p}^*$. For each $t \in \mathbb{R}$ and any $n \in \mathbb{N}$ there is a $k_n \in \mathbb{Z}$ such that $k_n t_n \leq t < (k_n+1)t_n$, so that $0 \leq t - k_n t_n < t_n$. Hence, given $\epsilon > 0$ there is a $\delta > 0$ such that if $t_n < \delta$, then $|\phi_{t-k_n t_n}(\boldsymbol{p}_n) - \boldsymbol{p}_n| < \epsilon/3$. By using the smoothness of the flow, one has that there is an $N_1 \geq 1$ such that if $n > N_1$, then $|\phi_t(\boldsymbol{p}_n) - \phi_t(\boldsymbol{p}^*)| < \epsilon/3$. Finally, there is an $N_2 \geq 1$ such that $|\boldsymbol{p}_n - \boldsymbol{p}^*| < \epsilon/3$ if $n \geq N_2$. Now, using the properties of the flow detailed in Lemma 4.2 one has that

$$\phi_t(\boldsymbol{p}_n) = \phi_t(\phi_{-t_n}(\boldsymbol{p}_n)) = \phi_t(\phi_{-k_n t_n}(\boldsymbol{p}_n)) = \phi_{t-k_n t_n}(\boldsymbol{p}_n).$$

Thus, for $N \geq \max\{N_1, N_2\}$ one has that

$$|\phi_t(\boldsymbol{p}^*) - \boldsymbol{p}^*| \leq |\phi_t(\boldsymbol{p}^*) - \phi_t(\boldsymbol{p}_n)| + |\phi_t(\boldsymbol{p}_n) - \boldsymbol{p}_n| + |\boldsymbol{p}_n - \boldsymbol{p}^*| < \epsilon.$$

This implies that $\phi_t(\boldsymbol{p}^*) = \boldsymbol{p}^*$ for all $t \in \mathbb{R}$, which means that $\boldsymbol{p}^*$ is a critical point. □

As seen in the next example, the result of Theorem 5.16 can be used to show that a system possesses no invariant Jordan curves.

Example. Consider the system

$$\dot{x}_1 = 1 + x_2^2, \quad \dot{x}_2 = x_1 x_2.$$

Since the system has no critical points, by Theorem 5.16 there exist no invariant Jordan curves. Note that Bendixson's criterion does not yield any information, as $\nabla \cdot \boldsymbol{f}(\boldsymbol{x}) = x_1$.

Now that it is known that periodic orbits must contain critical points in the interior, the answer to the question regarding the nature of the critical points can now be pursued. Rewrite equation (5.1) as the nonautonomous scalar equation

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}.$$

Set

$$\tan\theta := \frac{f_2(x_1,x_2)}{f_1(x_1,x_2)};$$

θ is the angle the vector field makes with the positive x_1-axis. Let $\gamma\subset\mathbb{R}^2$ be a positively oriented Jordan curve which does not contain any critical points of $\boldsymbol{f}$. The index of $\boldsymbol{f}$ with respect to γ is given by

$$j_{\boldsymbol{f}}(\gamma) := \frac{1}{2\pi}\oint_\gamma \mathrm{d}\theta = \frac{1}{2\pi}\oint_\gamma \frac{f_1\,\mathrm{d}f_2 - f_2\,\mathrm{d}f_1}{f_1^2+f_2^2}. \tag{5.2}$$

$j_{\boldsymbol{f}}(\gamma)$ represents the number of multiples of 2π the angle that $\boldsymbol{f}$ makes with the positive x_1-axis changes as γ is traversed once. One has that $j_{\boldsymbol{f}}(\gamma)$ varies continuously with any continuous deformation of γ which does not lead to encounters with critical points. As a consequence of [**15**, Theorem 3.12.1,Corollary 3.12.1] one has that:

Lemma 5.17. *Let $\gamma\subset\mathbb{R}^2$ be a positively oriented Jordan curve which does not contain any critical points of $\boldsymbol{f}$. Then*

(a) if $\mathrm{int}(\gamma)$ does not contain any critical points, then $j_{\boldsymbol{f}}(\gamma)=0$

(b) if γ_1 and γ_2 are Jordan curves with $\gamma_1\subset\mathrm{int}(\gamma_2)$, and if there are no critical points in $\mathrm{int}(\gamma_2)\cap\mathrm{ext}(\gamma_1)$, then $j_{\boldsymbol{f}}(\gamma_1)=j_{\boldsymbol{f}}(\gamma_2)$.

As a consequence of Lemma 5.17, one can define the index of a critical point $\boldsymbol{x}_0$ in the following manner. Let γ be a Jordan curve such that $\boldsymbol{x}_0\in\mathrm{int}(\gamma)$ and that $\mathrm{int}(\gamma)$ contains no other critical points of $\boldsymbol{f}$. Under this scenario set

$$j_{\boldsymbol{f}}(\boldsymbol{x}_0) := j_{\boldsymbol{f}}(\gamma).$$

If γ encloses a finite number of critical points, then a proper application of Lemma 5.17 (see [**15**, Theorem 3.12.2]) yields the following:

Lemma 5.18. *Let $\gamma\subset\mathbb{R}^2$ be a positively oriented Jordan curve whose interior contains the critical points $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n$. Then*

$$j_{\boldsymbol{f}}(\gamma) = \sum_{k=1}^n j_{\boldsymbol{f}}(\boldsymbol{x}_j).$$

It is now time to understand the manner in which one can compute $j_{\boldsymbol{f}}(\boldsymbol{x}_0)$, where $\boldsymbol{x}_0$ is an isolated critical point. Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be two vector fields such that $\boldsymbol{f}(\boldsymbol{x}_0)=\boldsymbol{g}(\boldsymbol{x}_0)=0$. Further assume that $\boldsymbol{g}(\boldsymbol{x})$ is a continuous deformation of $\boldsymbol{f}(\boldsymbol{x})$. For a given $\epsilon>0$ sufficiently small one has that there is a $\delta>0$ such that for $\gamma:=\partial B(\boldsymbol{x}_0,\delta)$ one has $|\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{g}(\boldsymbol{x})|<\epsilon$. Since $\boldsymbol{f}(\boldsymbol{x})\neq\boldsymbol{0}$ on γ, by making ϵ sufficiently small one can guarantee that $\boldsymbol{f}$ and $\boldsymbol{g}$ roughly point in the same direction all along γ. By definition this necessarily implies that

$$j_{\boldsymbol{f}}(\boldsymbol{x}_0) = j_{\boldsymbol{g}}(\boldsymbol{x}_0). \tag{5.3}$$

In other words, the index is unchanged relative to small perturbations of the vector field. This observation leads to the following result (see [**15**, Theorem 3.12.5]):

Lemma 5.19. *Consider*

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{r}(\boldsymbol{x}),$$

where $|\boldsymbol{r}(\boldsymbol{x})|=\mathcal{O}(|\boldsymbol{x}|^2)$. One has that

$$j_{\boldsymbol{A}\boldsymbol{x}}(\boldsymbol{0}) = j_{\boldsymbol{A}\boldsymbol{x}+\boldsymbol{r}(\boldsymbol{x})}(\boldsymbol{0}).$$

As a consequence of Lemma 5.19, in order to compute the index of a critical point it is sufficient to compute the index of the associated linearized problem. Using the definition in equation (5.2) yields that in general,

$$j_{\boldsymbol{A}\boldsymbol{x}}(\boldsymbol{0}) = \frac{\det(\boldsymbol{A})}{2\pi}\oint_\gamma \frac{x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1}{(a_{11}x_1+a_{12}x_2)^2+(a_{21}x_1+a_{22}x_2)^2}. \tag{5.4}$$

Henceforth assume that $\det(\boldsymbol{A}) \neq 0$. Now, it can be shown that the index is invariant under a nonsingular linear transformation; hence, when computing $j_{\boldsymbol{Ax}}(\boldsymbol{0})$ it is sufficient to consider those $\boldsymbol{A}$ which have the Jordan forms

$$\boldsymbol{A}_{\mathrm{r}} := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \boldsymbol{A}_{\mathrm{d}} := \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \quad \boldsymbol{A}_{\mathrm{c}} := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Assuming that $a \neq 0$ for $\boldsymbol{A}_{\mathrm{d}}$, one has that there is a continuous deformation such that either

$$\boldsymbol{A}_{\mathrm{d}} \mapsto \begin{pmatrix} a+\epsilon_1 & 0 \\ 0 & a+\epsilon_2 \end{pmatrix}, \quad \mathrm{sign}(a+\epsilon_1) = \mathrm{sign}(a+\epsilon_2) = \mathrm{sign}(a),$$

or

$$\boldsymbol{A}_{\mathrm{d}} \mapsto \begin{pmatrix} a+\epsilon & -b \\ b & a+\epsilon \end{pmatrix}, \quad \mathrm{sign}(a+\epsilon) = \mathrm{sign}(a),\ b \in \mathbb{R}^+.$$

As a consequence of the discussion leading to equation (5.3) one then has that $j_{\boldsymbol{A}_{\mathrm{d}}\boldsymbol{x}}(\boldsymbol{0}) = j_{\boldsymbol{A}_{\mathrm{r}}\boldsymbol{x}}(\boldsymbol{0})$ in the case that $\mathrm{sign}(b) = \mathrm{sign}(a)$, or $j_{\boldsymbol{A}_{\mathrm{d}}\boldsymbol{x}}(\boldsymbol{0}) = j_{\boldsymbol{A}_{\mathrm{c}}\boldsymbol{x}}(\boldsymbol{0})$. Hence, it is enough to compute the indices only in the cases of $\boldsymbol{A}_{\mathrm{r}}$ and $\boldsymbol{A}_{\mathrm{c}}$.

First consider $\boldsymbol{A}_{\mathrm{r}}$. Upon evaluating equation (5.4) over the ellipse

$$\gamma := \{(x_1, x_2) = \left(\frac{1}{a}\cos t, \frac{1}{b}\sin t\right) : 0 \leq t \leq 2\pi\},$$

and noting that the curve is positively oriented if $ab > 0$ and negatively oriented if $ab < 0$, one sees that

$$j_{\boldsymbol{A}_{\mathrm{r}}\boldsymbol{x}}(\boldsymbol{0}) = \begin{cases} -1, & ab < 0 \\ +1, & ab > 0. \end{cases}$$

Now consider $\boldsymbol{A}_{\mathrm{c}}$. Evaluating equation (5.4) over the positively oriented unit circle quickly yields that $j_{\boldsymbol{A}_{\mathrm{c}}\boldsymbol{x}}(\boldsymbol{0}) = +1$. The following result has now been proven:

Lemma 5.20. *Consider $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ under the condition that $\det(\boldsymbol{A}) \neq 0$. One has that $j_{\boldsymbol{Ax}}(\boldsymbol{0}) = -1$ if $\boldsymbol{0}$ is a saddle point; otherwise, $j_{\boldsymbol{Ax}}(\boldsymbol{0}) = +1$.*

As a consequence of Lemma 5.19 one has the following result concerning critical points of nonlinear systems.

Corollary 5.21. *Consider*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}),$$

where $\boldsymbol{f}(\boldsymbol{x}_0) = \boldsymbol{0}$. Assume that $\det(\mathrm{D}\boldsymbol{f}(\boldsymbol{x}_0)) \neq 0$. One has that

$$j_{\boldsymbol{f}}(\boldsymbol{x}_0) = \begin{cases} -1, & \boldsymbol{x}_0 \text{ is a saddle point} \\ +1, & \text{otherwise.} \end{cases}$$

Now suppose that γ is a Jordan curve which is invariant under the flow. It may be possible that γ contains critical points; henceforth, it will be assumed that there exist at most finitely many. The definition of $j_{\boldsymbol{f}}(\gamma)$ given in equation (5.2) requires that no critical points be on γ; however, this technical difficulty can be overcome [**15**, Remark 3.12.1]. The proof of the following result is that for [**15**, Theorem 3.12.3].

Theorem 5.22. *If γ be is a Jordan curve which is invariant under the flow, then $j_{\boldsymbol{f}}(\gamma) = +1$.*

By applying Corollary 5.21 to Theorem 5.22 one gets the following result.

Corollary 5.23. *If γ is an invariant Jordan curve which encloses only one critical point, then that point cannot be a saddle point.*

Proof: Suppose otherwise. By Theorem 5.22 one has that $j_{\boldsymbol{f}}(\gamma) = +1$, whereas by Corollary 5.21 one has that the index of a saddle point is -1. This is a contradiction, as the index is invariant under continuous deformation of γ. □

Example. Consider

$$\dot{x}_1 = x_1 + x_2, \quad \dot{x}_2 = x_1 - 2x_2 + x_1^3 + x_2^3.$$

The only critical point is $(0,0)$. This critical point is a saddle point, as the eigenvalues associated with the linearization are $\lambda = (-1 \pm \sqrt{13})/2$. Hence, there exists no periodic solution to the system. Note that Bendixson's criterion does not yield any information, as $\nabla \cdot f(x) = -1 + 3x_2^2$.

The proof of the following final result is left for the student.

Lemma 5.24. *Consider*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}).$$

Assume that for each critical point $\boldsymbol{x}_0$ one has that $\det(\mathrm{D}\boldsymbol{f}(\boldsymbol{x}_0)) \neq 0$. Let γ be a Jordan curve which is invariant under the flow. Then:

(a) $\mathrm{int}(\gamma)$ must contain an odd number of critical points

(b) of the $2n+1$ critical points contained in $\mathrm{int}(\gamma)$, n are saddle points.

5.4. Periodic vector fields

Now consider the following variant of equation (5.1):

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \quad \boldsymbol{f}(t+T, \boldsymbol{x}) = \boldsymbol{f}(t, \boldsymbol{x}), \tag{5.5}$$

where $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is smooth. As an autonomous first-order system this can be rewritten as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \quad \dot{t} = 1,$$

i.e., for $\boldsymbol{y} := (\boldsymbol{x}, t)^{\mathrm{T}}$,

$$\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y}), \quad \boldsymbol{g}(y) := (\boldsymbol{f}(\boldsymbol{y}), 1)^{\mathrm{T}}. \tag{5.6}$$

Recall the discussion on Poincaré maps for equation (5.6) leading to Definition 5.7. A Poincaré map for periodic vector fields of equation (5.5) can be defined in the following manner. For $\ell \in \mathbb{Z}$ set

$$\Sigma_\ell := \{(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^n \,:\, t = \ell T\},$$

and identify $\Sigma := \Sigma_0$ with Σ_1, so that Σ is a Poincaré section on the cylinder. The Poincaré map for equation (5.6) will be defined by $P(\boldsymbol{y}) := \phi_T(\boldsymbol{y})$. Equivalently, if $\boldsymbol{x}(t; \boldsymbol{x}_0)$ is the solution to equation (5.5) with the initial data $\boldsymbol{x}(0) = \boldsymbol{x}_0$, then one has $P(\boldsymbol{x}_0) = \boldsymbol{x}(T; \boldsymbol{x}_0)$. A definitive relationship between the solution and the iterates of the Poincaré map is given in the following result:

Lemma 5.25. *Set $P^k := P \circ P^{k-1}$ for $k = 1, 2, \dots$. Then $P^k(\boldsymbol{x}_0) = \boldsymbol{x}(kT; \boldsymbol{x}_0)$ for $k \in \mathbb{Z}$.*

Proof: Since $\boldsymbol{f}(t+T, \boldsymbol{x}) = \boldsymbol{f}(t, \boldsymbol{x})$, by a standard induction argument one has that $\boldsymbol{f}(t+kT, \boldsymbol{x}) = \boldsymbol{f}(t, \boldsymbol{x})$ for each $k \in \mathbb{Z}$. Set $\boldsymbol{z}(t) := \boldsymbol{x}(t+kT; \boldsymbol{x}_0)$. Then

$$\dot{\boldsymbol{z}} = \boldsymbol{f}(t, \boldsymbol{z}), \quad \boldsymbol{z}(0) = \boldsymbol{x}(kT; \boldsymbol{x}_0),$$

so by uniqueness one must have that $\boldsymbol{z}(t) = \boldsymbol{x}(t; \boldsymbol{x}(kT; \boldsymbol{x}_0))$. In other words, $\boldsymbol{x}(t+kT; \boldsymbol{x}_0) = \boldsymbol{x}(t; \boldsymbol{x}(kT; \boldsymbol{x}_0))$. Now, $P(\boldsymbol{x}_0) = \boldsymbol{x}(T; \boldsymbol{x}_0)$, and

$$P^2(\boldsymbol{x}_0) = P \circ P(\boldsymbol{x}_0) = \boldsymbol{x}(T; \boldsymbol{x}(T; \boldsymbol{x}_0)) = \boldsymbol{x}(2T; \boldsymbol{x}_0).$$

The rest of the proof follows from an induction argument. $\square$

Corollary 5.26. *The solution $\boldsymbol{x}(t; \boldsymbol{x}_0)$ to equation (5.4) is kT-periodic if and only if $P^k(\boldsymbol{x}_0) = \boldsymbol{x}_0$.*

Proof: Suppose that $P^k(\boldsymbol{x}_0) = \boldsymbol{x}_0$. By Lemma 5.25 this implies that $\boldsymbol{x}(kT; \boldsymbol{x}_0) = \boldsymbol{x}_0$, so that $\boldsymbol{x}(t+kT; \boldsymbol{x}_0) = \boldsymbol{x}(t; \boldsymbol{x}(kT; \boldsymbol{x}_0)) = \boldsymbol{x}(t; \boldsymbol{x}_0)$. Thus, the solution is kT-periodic. The other direction follows by the definition of the Poincaré map and Lemma 5.25. $\square$

As a consequence of Corollary 5.26, one can discuss periodic orbits for the Poincaré map.

Definition 5.27. The point $\boldsymbol{p}$ is a periodic point of period N if $P^N(\boldsymbol{p}) = \boldsymbol{p}$, but $P^k(\boldsymbol{p}) \neq \boldsymbol{p}$ for $k = 1, \dots, N-1$. The periodic orbit for the map is given by $\{\boldsymbol{p}, P(\boldsymbol{p}), \dots, P^{N-1}(\boldsymbol{p})\}$.

Example. For a simple example, consider the damped and periodically forced harmonic oscillator modeled by

$$\ddot{x} + 2\mu\dot{x} + x = h\cos\omega t,$$

where $\mu \in [0, 1)$ and $h, \omega \in \mathbb{R}^+$ (also see [**17**, Example 5.4]). If $\mu = 0$, it will be assumed that $\omega \neq 1$ (i.e., there will be no resonant forcing). The solution to the initial value problem is given by

$$x(t) = c_1\mathrm{e}^{-\mu t}\cos(\sqrt{1-\mu^2}\,t) + c_2\mathrm{e}^{-\mu t}\sin(\sqrt{1-\mu^2}\,t) + A\cos\omega t + B\sin\omega t,$$

where

$$A := \frac{1-\omega^2}{4\mu^2\omega^2 + (1-\omega^2)^2}h, \quad B := \frac{2\mu\omega}{4\mu^2\omega^2 + (1-\omega^2)^2}h,$$

and

$$c_1 := x(0) - A, \quad c_2 := \frac{\dot{x}(0) + \mu x(0) - A\mu - B\omega}{\sqrt{1-\mu^2}}.$$

For $\boldsymbol{x} := (x, \dot{x})^{\mathrm{T}}$ and $T := 2\pi/\omega$ the Poincaré map is given by $P(\boldsymbol{x}(0)) = \boldsymbol{x}(T)$, i.e.,

$$P(\boldsymbol{x}(0)) = \begin{pmatrix} c_1\mathrm{e}^{-\mu T}\cos\gamma + c_2\mathrm{e}^{-\mu T}\sin\gamma + A \\ (-c_1\mu + c_2\sqrt{1-\mu^2})\mathrm{e}^{-\mu T}\cos\gamma + (c_1\sqrt{1-\mu^2} + c_2\mu)\mathrm{e}^{-\mu T}\sin\gamma + B\omega \end{pmatrix},$$

where

$$\gamma := \sqrt{1-\mu^2}\,T.$$

First suppose that $\mu > 0$. The unique fixed point is then given by $\boldsymbol{x}^* := (A, B\omega)^{\mathrm{T}}$. Furthermore, it is not difficult to show that

$$\lim_{n\to+\infty} P^n(\boldsymbol{x}(0)) = \boldsymbol{x}^*;$$

hence, the periodic solution is asymptotically stable. Now suppose that $\mu = 0$. One then has that

$$A = \frac{h}{1-\omega^2}, \quad B = 0, \quad c_1 = x(0) - A, \quad c_2 = \dot{x}(0),$$

so that the Poincaré map satisfies

$$P(\boldsymbol{x}(0)) = \begin{pmatrix} \cos T & \sin T \\ -\sin T & -cos T \end{pmatrix}\boldsymbol{x}(0) + A\begin{pmatrix} 1-\cos T \\ \sin T \end{pmatrix}.$$

If $\omega \notin \mathbb{Q}$, then the unique stable fixed point is given by $\boldsymbol{x}^* := (A, 0)^{\mathrm{T}}$. Furthermore, as long as $\omega \notin \mathbb{Q}$, then P^ℓ has a unique fixed point $\boldsymbol{x}^*$ for each $\ell \in \mathbb{N}$. If $\omega \in \mathbb{Q}\backslash\{1\}$, then there is still a unique stable fixed point for P; however, $P^\ell = \mathbb{1}$ for some $\ell \in \mathbb{N}\backslash\{1\}$, so that in this case all solutions are $2\ell\pi$-periodic.

Consider the general problem of finding fixed points for P. If there is a closed ball $B \subset \Sigma$ such that $P(\partial B) \subset B$, then as an application of Theorem 5.15 one gets the existence of a fixed point of P, i.e., a point $\boldsymbol{x}^* \in B$ such that $P(\boldsymbol{x}^*) = \boldsymbol{x}^*$. By Corollary 5.26 this in turn implies the existence of a periodic orbit $\boldsymbol{x}(t; \boldsymbol{x}^*)$. In general, not much more can be said about the Poincaré map. However, this is not true in the case of scalar vector fields. In particular, one can use the uniqueness of solutions to determine the behavior of the sequence $\{P^n(x_0)\}_{n=0}^\infty$.

Proposition 5.28. *Suppose in equation (5.5) that $f : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$. Then $\{P^n(x_0)\}_{n=0}^\infty$ is a monotone sequence.*

Proof: Without loss of generality suppose that $P(x_0) > x_0$. Define the sequence of solutions $z_k : [0, T] \mapsto \mathbb{R}$ by $z_k(t) := x(t; P^k(x_0))$. Note that $z_k(T) = P^{k+1}(x_0)$. By the uniqueness of solutions, $z_1(t) > z_0(t)$ for all $t \in [0, T]$; hence, $P^2(x_0) > P(x_0)$. An induction argument yields that $P^{n+1}(x_0) > P^n(x_0)$ for all $n \in \mathbb{N}$. $\square$

Remark 5.29. If $P(x_0) < x_0$, then the sequence is monotone decreasing.

An application of Proposition 5.28 yields a Poincaré-Bendixson-type theorem for scalar periodic vector fields.

Theorem 5.30. *Consider equation (5.5) in the case that $f : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$. If the solution $x(t; x_0)$ is uniformly bounded, then there exists a T-periodic solution.*

Proof: By Proposition 5.28 the sequence $\{P^n(x_0)\}$ is monotone. By supposition, this sequence is bounded; hence, there exists a $y \in \mathbb{R}$ such that $P^n(x_0) \to y$ as $n \to +\infty$. Now, by continuity one has that

$$x(T; y) = \lim_{n\to\infty} x(T; P^n(x_0)) = \lim_{n\to\infty} x(0; P^{n+1}(x_0)) = x(0; \lim_{n\to\infty} P^{n+1}(x_0)) = x(0; y);$$

hence, $x(t + T; y) = x(t; y)$ for all $t \geq 0$. □

Given the existence of fixed points for the Poincaré map, one defines stability as below.

Definition 5.31. p is a stable fixed point of P if for each $\epsilon > 0$ there is a $\delta > 0$ such that if $|x - p| < \delta$, then $|P^n(x) - p| < \epsilon$ for all $n \in \mathbb{N}$. Otherwise, the fixed point is unstable. The fixed point is asymptotically stable if it is stable and $P^n(x) \to p$ as $n \to +\infty$.

Example. Demonstrate the graphical iteration of scalar maps.

Theorem 5.32. *Let $P : \mathbb{R} \mapsto \mathbb{R}$ be a C^1 map. A fixed point p is asymptotically stable if $|P'(p)| < 1$, and unstable if $|P'(p) > 1$.*

Proof: Upon setting $u := x-p$ and $g(u) := P(u+p)-P(p)$, the map $x_{n+1} = P(x_n)$ becomes $u_{n+1} = g(u_n)$. Noting that $g(0) = 0$, one then has that without loss of generality, the fixed point is $p = 0$.

Since $P(0) = 0$, by the Fundamental Theorem of Calculus one has that $P(x) = \int_0^x P'(s)\,\mathrm{d}s$. Given $\epsilon > 0$, set

$$m_\epsilon := \min_{|x|\leq\epsilon} |P'(x)|, \quad M_\epsilon := \max_{|x|\leq\epsilon} |P'(x)|;$$

thus, for $|x| \leq \epsilon$ one has $m_\epsilon|x| \leq |P(x)| \leq M_\epsilon|x|$. Upon repeated applications of the chain rule one can then show that

$$m_\epsilon^n|x| \leq |P^n(x)| \leq M_\epsilon^n|x|.$$

Suppose that $|P'(0)| < 1$, so that for $\epsilon > 0$ sufficiently small one has that $M_\epsilon < 1$. Then for $\delta = \epsilon/M_\epsilon$ and $|x| < \delta$ one has that $|P^n(x)| < \epsilon$, so that the fixed point is stable. Furthermore, since

$$\lim_{n\to+\infty} |P^n(x)| \leq \lim_{n\to+\infty} M_\epsilon^n|x| = 0,$$

the fixed point is asymptotically stable.

Now suppose that $|P'(0)| > 1$. For $\epsilon > 0$ sufficiently small one then has that $m_\epsilon > 1$, so that $|P^n(x)| \geq m_\epsilon^n|x|$ as long as $|x| \leq \epsilon$. Given an $\epsilon_0 < \epsilon$ and x_0 with $|x_0| < \epsilon_0$, there is an N such that $|P^N(x_0)| \geq m_\epsilon^N|x_0| > \epsilon_0$. Since x_0 is arbitrary, the fixed point is unstable. □

It is now necessary to understand how one can compute the derivative of the Poincaré map at a fixed point. Let $\gamma(t)$ be a T-periodic solution such that $\gamma(0) = p$. Let the solution to equation (5.5) be denoted $x(t; x_0)$. Since $P(x_0) = x(T; x_0)$, we have that

$$P'(x_0) = \frac{\mathrm{d}}{\mathrm{d}x_0} x(T; x_0).$$

Upon using the chain rule and smoothness one gets that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}x_0} x\right) = f_x(t, x)\frac{\mathrm{d}}{\mathrm{d}x_0} x.$$

The equation is linear, and since

$$\frac{\mathrm{d}}{\mathrm{d}x_0} x(0; x_0) = 1,$$

it has the solution

$$\frac{\mathrm{d}}{\mathrm{d}x_0}x(t;x_0) = \exp\left(\int_0^t f_x(s, x(s;x_0))\,\mathrm{d}s\right).$$

Evaluating at $t = T$ and $x_0 = p$ yields the following result.

Lemma 5.33. *Consider equation (5.5) in the case that $f : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$. Let $\gamma(t)$ be a T-periodic solution such that $\gamma(0) = p$. Then*

$$P'(p) = \mathrm{e}^{a_0}, \quad a_0 := \int_0^T f_x(t, \gamma(t))\,\mathrm{d}t.$$

Note that $P'(p) > 0$ for any fixed point p. As an application of Theorem 5.32 one has the following result.

Corollary 5.34. *If $a_0 < 0$, then the fixed point is asymptotically stable, whereas if $a_0 > 0$, then the fixed point is unstable.*

Example. Consider equation (5.5) in the case that $f(t, x) = -x^3 + a(t)$, where $a(t + T) = a(t)$. It will be shown that for this vector field that there is a unique asymptotically stable T-periodic solution.

First suppose that $P(p) = p$ for some p, and let $\gamma(t)$ be the corresponding T-periodic solution. Since $a_0 = \int_0^T -3\gamma^2(t)\,\mathrm{d}t < 0$, by Corollary 5.34 the fixed point is asymptotically stable. In order to show that the fixed point is unique, set $g(x) := x - P(x)$. Note that at a fixed point q, i.e., $g(q) = 0$, one has that $g'(q) = 1 - P'(q) > 0$. Let $x_1 < x_2$ be two fixed points such that $g(x) \neq 0$ for $x_1 < x < x_2$. Since $g'(x_i) > 0$ for $i = 1, 2$, by continuity there must exist a point $x_3 \in (x_1, x_2)$ such that $g(x_3) = 0$ with $g'(x_3) \le 0$. This is a contradiction; hence, there can exist at most one fixed point.

It is now time to show that the assumed fixed point, which is unique, actually exists. Since $a(t)$ is continuous and periodic, there exists an $M > 0$ such that $|a(t)| \le M$ for all $t \in [0, T]$. Set

$$U_+ := \{(t, x) : -x^3 - M > 0\}, \quad U_- := \{(t, x) : -x^3 + M < 0\}.$$

If $x \in U_+$, then $\dot{x} > 0$, while if $x \in U_-$, then $\dot{x} < 0$. Thus, if $x \in U_+$ one has that $P(x) > x$, while if $x \in U_-$ one has $P(x) < x$. Since the Poincaré map is continuous, there is a point p such that $P(p) = p$.

6. Applications of center manifold theory

Again consider the autonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu), \tag{6.1}$$

where $\boldsymbol{f}\ \mathbb{R}^n \times \mathbb{R}^k \mapsto \mathbb{R}^n$ is smooth and satisfies $\boldsymbol{f}(\boldsymbol{0}, 0) = \boldsymbol{0}$. Recall the statements of the manifold theorems given in Section 3. While it is not explicitly stated therein, it can be shown that these manifolds vary smoothly with respect to parameters. Now suppose that $\dim(E^{\mathrm{c}}) = \ell$, where $1 \leq \ell \leq n$. As an application of the Center Manifold Theorem 3.6 there exists an invariant $(\ell + k)$-dimensional invariant manifold, so that the governing equations are of dimension $\ell + k$. This reduced set of equations will determine all of the interesting solutions near $\boldsymbol{x} = \boldsymbol{0}$, as it is known that any solutions on $W^{\mathrm{s}}\,(W^{\mathrm{u}})$ will have exponential behavior as $t \to +\infty\, (t \to -\infty)$.

The manifold theorems are also important in that they implicitly tell us how to compute the flow on W^{c}. Consider equation (6.1) written in the form

$$\dot{\boldsymbol{y}} = \boldsymbol{A}(\mu)\boldsymbol{y} + \boldsymbol{g}_1(\boldsymbol{y}, \boldsymbol{z}, \mu), \quad \dot{\boldsymbol{z}} = \boldsymbol{B}(\mu)\boldsymbol{z} + \boldsymbol{g}_2(\boldsymbol{y}, \boldsymbol{z}, \mu), \tag{6.2}$$

where $\boldsymbol{y} \in E^{\mathrm{s}} \oplus E^{\mathrm{u}}$, $\boldsymbol{z} \in E^{\mathrm{c}}$, and $|\boldsymbol{g}_j(\boldsymbol{y}, \boldsymbol{z}, \mu)| = \mathcal{O}(2)$, where $\mathcal{O}(m)$ implies the inclusion of all terms of the form $|\boldsymbol{y}|^i |\boldsymbol{z}|^j |\mu|^k$ with $i + j + k \geq m$. As a consequence of the Center Manifold Theorem 3.6 it is known that W^{c} is given by the graph $\boldsymbol{y} = h^{\mathrm{c}}(\boldsymbol{z}, \mu)$, where $\mathrm{R}(\mathrm{D}h^{\mathrm{c}}(\boldsymbol{0}, 0)) = E^{\mathrm{c}}$. The local flow on W^{c} is then given by

$$\dot{\boldsymbol{z}} = \boldsymbol{B}(\mu)\boldsymbol{z} + \boldsymbol{g}_2(h^{\mathrm{c}}(\boldsymbol{z}, \mu), \boldsymbol{z}, \mu). \tag{6.3}$$

Since $h^{\mathrm{c}}(\boldsymbol{z}, \mu)$ is smooth, by using the invariance property of the manifold it can be computed via a Taylor expansion.

6.1. *Reduction to scalar systems*

As a first example, consider the case that $\mu \in \mathbb{R}$ and that

$$\mathrm{D}\boldsymbol{f}(\boldsymbol{0}, 0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.4}$$

In this case there is a one-dimensional stable manifold, and a one-dimensional center manifold; furthermore, the stable manifold is tangent to $\mathrm{span}\{\boldsymbol{e}_1\}$, and the center manifold is tangent to $\mathrm{span}\{\boldsymbol{e}_2\}$. It can be shown via the theory of *normal forms* (e.g., see [**18**, Chapter 19]) that equation (6.2) can be written as

$$\begin{aligned} \dot{x} &= -(1 + a_1\mu)x + b_1 xy + \mathcal{O}(3) \\ \dot{y} &= a_2\mu + a_3\mu y + b_2 y^2 + \mathcal{O}(3), \end{aligned} \tag{6.5}$$

where the constants $a_j, b_j \in \mathbb{R}$. Henceforth ignore the $\mathcal{O}(3)$ terms, as an application of the Implicit Function Theorem makes them irrelevant. For the truncated equation (6.5) the line $x = 0$ is invariant; hence, it is the center manifold. The flow on the center manifold is then governed by

$$\dot{y} = a_2\mu + a_3\mu y + b_2 y^2. \tag{6.6}$$

Assuming that $b_2 \neq 0$, rescale equation (6.6) via $s := |b_2|t$, so that equation (6.6) becomes

$$y' = \alpha_2\mu + \alpha_3\mu y + \delta y^2, \quad {}' := \frac{\mathrm{d}}{\mathrm{d}s}, \tag{6.7}$$

where $\delta \in \{-1, +1\}$ and $\alpha_j = a_j/|b_2|$. The analysis of equation (6.7) is straightforward. If $\alpha_2 \neq 0$, the critical points are given by

$$y = \pm\sqrt{-\delta\alpha_2\mu} + \mathcal{O}(|\mu|).$$

This is an example of a *saddle-node bifurcation* [**18**, Chapter 20.1c]. Note that the expression makes sense if and only if $\delta\alpha_2\mu \in \mathbb{R}^-$. The flow on the center manifold is depicted in Figure 8, and the full flow near the origin is depicted in Figure 9. If $\alpha_2 = 0$, the critical points are given by

$$y = 0, \quad y = -\delta\alpha_3\mu.$$

This is an example of a *transcritical bifurcation* [**18**, Chapter 20.1d]. The student is invited to draw the bifurcation diagrams similar to those in Figure 8 and Figure 9 for the saddle-node bifurcation.

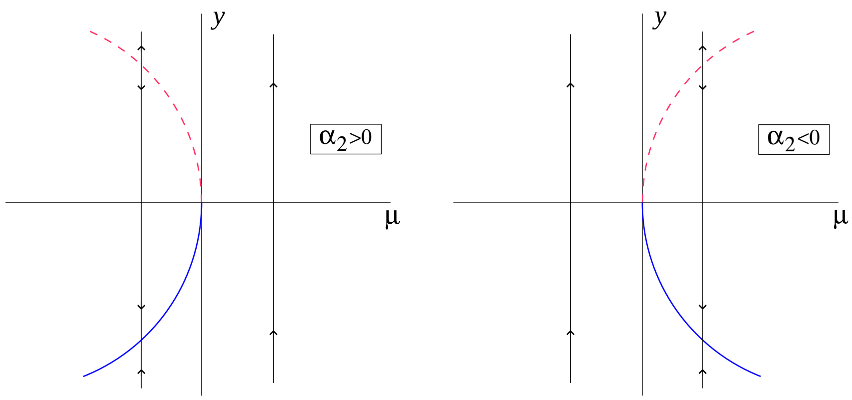

Figure 8: The bifurcation diagrams for equation (6.7) in the case that $\alpha_2 \neq 0$ and $\delta = +1$.

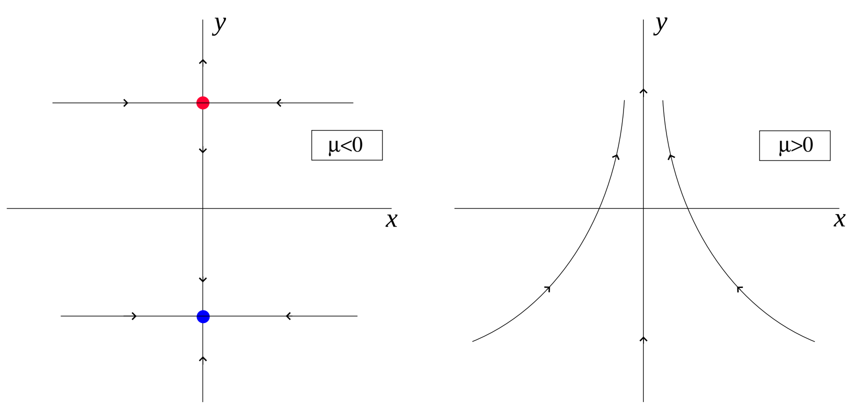

Figure 9: The flow near the origin for equation (6.7) in the case that $\delta = +1$ and $\alpha_2 > 0$.

Before considering the next example, the following result is needed.

Proposition 6.1. *Suppose that* $\boldsymbol{g} : \mathbb{R}^n \mapsto \mathbb{R}^n$ *is* C^2 *in a neighborhood of* $\boldsymbol{x} = \boldsymbol{0}$*, and suppose that* $\boldsymbol{g}(0, x_2, \dots, x_n) = \boldsymbol{0}$ *for all* $(0, x_2, \dots, x_n)$ *in a neighborhood of the origin. There exists a neighborhood* M *of the origin and a* $\boldsymbol{g}_1 \in C^1(M)$ *such that* $\boldsymbol{g}(\boldsymbol{x}) = x_1 \boldsymbol{g}_1(\boldsymbol{x})$*.*

Proof: By Taylor's theorem one has that

$$\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}(0, x_2, \dots, x_n) + \int_0^1 \frac{\partial}{\partial t} \boldsymbol{g}(tx_1, x_2, \dots, x_n)\, \mathrm{d}t.$$

Choose M so that the line between $\boldsymbol{x}$ and $(0, x_2, \dots, x_n)^{\mathrm{T}}$ lies in M. This yields that

$$\frac{\partial}{\partial t}\boldsymbol{g}(tx_1, x_2, \dots, x_n) = x_1 \mathrm{D}\boldsymbol{g}(tx_1, x_2, \dots, x_n)\boldsymbol{e}_1,$$

which in turn implies that

$$\boldsymbol{g}_1(\boldsymbol{x}) = \int_0^1 \mathrm{D}\boldsymbol{g}(tx_1, x_2, \dots, x_n)\boldsymbol{e}_1 \, \mathrm{d}t \in C^1(M). \qquad \square$$

For the next example, consider the system

$$\begin{aligned} \dot{x} &= -x + x^2 - y^2 \\ \dot{y} &= \epsilon y + xy - y^3. \end{aligned} \tag{6.8}$$

Note that the critical parameter b_2 present in the normal form of equation (6.5) has been set to zero in equation (6.8). At the critical point $(x, y, \epsilon) = (0, 0, 0)$ one has that

$$\sigma^{\mathrm{s}}(\mathrm{D}\boldsymbol{f}(\boldsymbol{0})) = \{-1\},\ E^{\mathrm{s}} = \mathrm{span}\{\boldsymbol{e}_1\}; \quad \sigma^{\mathrm{c}}(\mathrm{D}\boldsymbol{f}(\boldsymbol{0})) = \{0\},\ E^{\mathrm{c}} = \mathrm{span}\{\boldsymbol{e}_2\}.$$

Upon applying the Center Manifold Theorem 3.6 one knows that the center manifold is locally given by

$$x = h(y, \epsilon); \quad h(0, 0) = h_y(0, 0) = h_\epsilon(0, 0) = 0, \tag{6.9}$$

and the flow on W^{c} is given by

$$\dot{y} = \epsilon y + yh(y, \epsilon) - y^3. \tag{6.10}$$

The function $h(y, \epsilon)$ must now be determined. It is clear that $(0, 0, \epsilon)$ is a critical point for any $\epsilon \in \mathbb{R}$; hence, $h(0, \epsilon) \equiv 0$, so by Proposition 6.1 one can write $h(y, \epsilon) = yh_1(y, \epsilon)$. As a consequence of the smoothness of the vector field the function h_1 has a Taylor expansion, which by equation (6.9) is given by

$$h_1(y, \epsilon) = ay + b\epsilon + \mathcal{O}(2).$$

Since W^{c} is invariant one has that

$$\dot{x} = \frac{\partial h}{\partial y}\dot{y} + \frac{\partial h}{\partial \epsilon}\dot{\epsilon},$$

which yields that

$$-h(y, \epsilon) + h(y, \epsilon)^2 - y^2 = (2ay + b\epsilon + \mathcal{O}(2))(\epsilon y + yh(y, \epsilon) - y^3.$$

Simplifying the above expression gives

$$-(a + 1)y^2 - b\epsilon y + \mathcal{O}(3) = \mathcal{O}(3),$$

which necessarily implies that

$$a = -1, \quad b = 0.$$

In conclusion,

$$h(y, \epsilon) = y(-y + \mathcal{O}(2)). \tag{6.11}$$

Substituting the result of equation (6.11) into equation (6.10) yields that the flow on W^{c} is given by

$$\dot{y} = y(\epsilon - 2y^2 + \mathcal{O}(3)) \tag{6.12}$$

Set

$$m(y, \epsilon) := \epsilon - 2y^2 + \mathcal{O}(3).$$

Since $m(0, 0) = 0$ and $m_\epsilon(0, 0) = 1$, by the Implicit Function Theorem there exists an $\epsilon = \epsilon(y)$ and $y_0 > 0$ such that $m_\epsilon(y, \epsilon(y)) \equiv 0$ for all $|y| < y_0$. By inspection one has that $\epsilon(y) = 2y^2 + \mathcal{O}(3)$. The above example yields what is known as a *pitchfork bifurcation* [**18**, Chapter 20.1e]. The student is invited to draw the bifurcation diagrams for this problem similar to those in Figure 8 and Figure 9.

6.1.1. Example: singular perturbations

Consider the system

$$\begin{aligned} \dot{x} &= -x + \mu y + xy \\ \epsilon \dot{y} &= x - y - xy, \end{aligned} \tag{6.13}$$

where $\mu \in (0,1)$ and $0 < \epsilon \ll 1$. The system arises as a model of the kinetics of enzyme reactions (see [3, Example 1.4.3] and the references therein). Setting $s := \epsilon t$ and $z := x - y$ transforms equation (6.13) to the system

$$\begin{aligned} x' &= \epsilon f(x,z) \\ z' &= -z + x^2 - xz + \epsilon f(x,z), \end{aligned} \tag{6.14}$$

where $' := \mathrm{d}/\mathrm{d}s$ and

$$f(x,z) := -x + (x+\mu)(x-z).$$

Upon applying the Center Manifold Theorem 3.6 one knows that the center manifold for equation (6.14) is locally given by

$$z = h(x,\epsilon); \quad h(0,0) = h_x(0,0) = h_\epsilon(0,0) = 0,$$

and the flow on W^{c} is given by

$$x' = \epsilon f(x, h(x,\epsilon)). \tag{6.15}$$

The function $h(x,\epsilon)$ must now be determined. As a consequence of the smoothness of the vector field the function h has a Taylor expansion which is given by

$$h(x,\epsilon) = a\epsilon^2 + b\epsilon x + cx^2 + \mathcal{O}(3).$$

Since W^{c} is invariant one has that

$$z' = \frac{\partial h}{\partial x} x' + \frac{\partial h}{\partial \epsilon}\epsilon' = \mathcal{O}(3),$$

which eventually yields that

$$\mathcal{O}(3) = -a\epsilon^2 + (\mu - 1 - b)\epsilon x + (1-c)x^2 + \mathcal{O}(3) = \mathcal{O}(3).$$

This necessarily implies that

$$a = 0, \quad b = -(1-\mu), \quad c = 1,$$

so that one can conclude that

$$h(x,\epsilon) = x^2 - (1-\mu)\epsilon x + \mathcal{O}(3). \tag{6.16}$$

Substituting the result of equation (6.16) into equation (6.15) yields that the flow on W^{c} is given by

$$x' = \epsilon\left[-(1-\mu)\epsilon x + x^2 + \mathcal{O}(3)\right]; \tag{6.17}$$

hence, there is a transcritical bifurcation. The argument for the irrelevance of the $\mathcal{O}(4)$ terms is the same as for the previous example. The student is invited to draw the bifurcation diagrams for this problem similar to those in Figure 8 and Figure 9.

6.1.2. Example: hyperbolic conservation laws

A viscous conservation law is given by

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = \boldsymbol{u}_{xx}, \tag{6.18}$$

where $\boldsymbol{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is C^∞. A thorough discussion of conservation laws and their importance in applications can be found in [**16**, Part III]. The goal here is to find travelling waves, which are solutions $\boldsymbol{u}(z)$, $z := x - st$, of equation (6.18) which satisfy the asymptotics

$$\boldsymbol{u}(z) \to \begin{cases} \boldsymbol{u}_{\mathrm{L}}, & z \to -\infty \\ \boldsymbol{u}_{\mathrm{R}}, & z \to +\infty. \end{cases} \tag{6.19}$$

In the travelling frame equation (6.18) is written as

$$\boldsymbol{u}_t - s\boldsymbol{u}_z + \boldsymbol{f}(\boldsymbol{u})_z = \boldsymbol{u}_{zz}, \tag{6.20}$$

and the travelling wave will now be a steady-state solution, i.e., a solution to

$$-s\boldsymbol{u}_z + \boldsymbol{f}(\boldsymbol{u})_z = \boldsymbol{u}_{zz}. \tag{6.21}$$

It will be realized as a heteroclinic orbit. The following assumption will be required.

Assumption 6.2. The function $\boldsymbol{f}$ satisfies:

(a) for all $\boldsymbol{u} \in \mathbb{R}^n$, $\mathrm{D}\boldsymbol{f}(\boldsymbol{u})$ has distinct real eigenvalues

$$\lambda_1(\boldsymbol{u}) < \lambda_2(\boldsymbol{u}) < \cdots < \lambda_n(\boldsymbol{u})$$

(i.e., the system is *strictly hyperbolic*)

(b) $\langle \nabla\lambda_j(\boldsymbol{u}), \boldsymbol{r}_j(u)\rangle < 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$, where $\boldsymbol{r}_j(u)$ is the eigenvector associated with the eigenvalue $\lambda_j(\boldsymbol{u})$ (i.e., the system is *genuinely nonlinear*).

Remark 6.3. If $n = 1$, Assumption 6.2(b) is equivalent to specifying that $f(u)$ is convex. Furthermore, it guarantees that for each $s < f'(u_{\mathrm{L}})$ there is a unique $u_{\mathrm{R}}(s) > u_{\mathrm{L}}$ such that there is a solution to equation (6.21) which satisfies equation (6.19).

Lemma 6.4. *For each $\boldsymbol{u}_{\mathrm{L}} \in \mathbb{R}^n$ and each $1 \le k \le n$ there exists a curve of states $\boldsymbol{u}_{\mathrm{R}}^k(\rho)$ for $0 < \rho < \rho_0$ such that a travelling wave exists with speed $s = s^k(\rho)$. Furthermore,*

(a) $\boldsymbol{u}_{\mathrm{R}}^k(\rho)$ and $s^k(\rho)$ are C^r for any $r \in \mathbb{N}$, and

$$\lim_{\rho\to 0^+} \boldsymbol{u}_{\mathrm{R}}^k(\rho) = \boldsymbol{u}_{\mathrm{L}}, \quad \lim_{\rho\to 0^+} s^k(\rho) = \lambda_k(\boldsymbol{u}_{\mathrm{L}})$$

(b) $\lambda_k(\boldsymbol{u}_{\mathrm{R}}^k(\rho)) < s^k(\rho) < \lambda_k(\boldsymbol{u}_{\mathrm{L}})$

(c) $\lambda_{k-1}(\boldsymbol{u}_{\mathrm{L}}) < s^k(\rho) < \lambda_{k+1}(\boldsymbol{u}_{\mathrm{R}}^k(\rho))$.

Remark 6.5. Conditions (b) and (c) are known as the *Lax entropy inequalities.*

Proof: Set $\dot{} := \mathrm{d}/\mathrm{d}z$. Integrating equation (6.21) from $-\infty$ to z and using the fact that $\boldsymbol{u}(z) \to \boldsymbol{u}_{\mathrm{L}}$ as $z \to -\infty$ yields

$$\dot{\boldsymbol{u}} = \boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{u}_{\mathrm{L}}) - s(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{L}}).$$

Linearizing at the critical point $\boldsymbol{u}_{\mathrm{L}}$ yields $\boldsymbol{A} := \mathrm{D}\boldsymbol{f}(\boldsymbol{u}_{\mathrm{L}}) - s\mathbb{1}$. Upon noting that

$$\sigma(\boldsymbol{A}) = \{\lambda - s : \lambda \in \sigma(\mathrm{D}\boldsymbol{f}(\boldsymbol{u}_{\mathrm{L}}))\},$$

it is seen that a bifurcation can occur only if $s = \lambda_k(\boldsymbol{u}_{\mathrm{L}})$ for some $k = 1, \ldots, n$. Note that by Assumption 6.2(a) the equation describing the bifurcation will necessarily be a scalar ODE. A branch of solutions will be obtained for each k.

Setting $s_0 := \lambda_k(\boldsymbol{u}_{\mathrm{L}})$ and linearizing at the point $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{L}}$ yields

$$E^{\mathrm{c}} = \mathrm{span}\{\boldsymbol{r}_k(\boldsymbol{u}_{\mathrm{L}})\}.$$

The equations on W^{c} must now be computed. The graph of W^{c} is given by

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{L}} + \eta \boldsymbol{r}_k(\boldsymbol{u}_{\mathrm{L}}) + W(\eta, s), \tag{6.22}$$

where $W(\eta, s)$ is the complementary direction, i.e.,

$$W(\eta, s) = \sum_{j \neq k} a_j(\eta, s) \boldsymbol{r}_j(\boldsymbol{u}_{\mathrm{L}}).$$

As a consequence of the center manifold theorem one has that

$$a_j(0, s_0) = \mathrm{D}_s a_j(0, s_0) = 0, \quad \mathrm{D}_\eta a_j(0, s_0) = 0.$$

Let $\boldsymbol{l}_j(u)$ be the eigenvectors of $\mathrm{D}\boldsymbol{f}(\boldsymbol{u})^{\mathrm{T}}$ which satisfy

$$\langle \boldsymbol{l}_i(\boldsymbol{u}), \boldsymbol{r}_j(\boldsymbol{u}) \rangle = \delta_{ij}.$$

Upon taking a Taylor expansion for $\boldsymbol{f}(\boldsymbol{u})$ at $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{L}}$ and applying the operator $\langle \boldsymbol{l}_k(\boldsymbol{u}_{\mathrm{L}}), \cdot \rangle$ one sees that

$$\begin{aligned} \langle \boldsymbol{l}_k(\boldsymbol{u}_{\mathrm{L}}), (\boldsymbol{u} \dot{-} \boldsymbol{u}_{\mathrm{L}}) \rangle &= \langle \boldsymbol{l}_k(\boldsymbol{u}_{\mathrm{L}}), \mathrm{D}\boldsymbol{f}(\boldsymbol{u}_{\mathrm{L}})(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{L}}) \rangle - \langle \boldsymbol{l}_k(\boldsymbol{u}_{\mathrm{L}}), s(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{L}}) \rangle \\ &\quad + \langle \boldsymbol{l}_k(\boldsymbol{u}_{\mathrm{L}}), \frac{1}{2}\mathrm{D}^2\boldsymbol{f}(\boldsymbol{u}_{\mathrm{L}})(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{L}})^2 \rangle + \cdots . \end{aligned}$$

As a consequence of equation (6.22) and the fact that $\langle \ell_k(u_{\mathrm{L}}), W(\eta, s) \rangle = 0$, and since

$$(\mathrm{D}\boldsymbol{f}(\boldsymbol{u}_{\mathrm{L}}) - s\mathbb{1})(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{L}}) = \eta(s_0 - s)\boldsymbol{r}_k(\boldsymbol{u}_{\mathrm{L}}) + \sum_{j \neq k} a_j(\eta, s)(\lambda_j(\boldsymbol{u}_{\mathrm{L}}) - s)\boldsymbol{r}_j(\boldsymbol{u}_{\mathrm{L}}),$$

one sees that the flow on W^{c} is given by

$$\dot{\eta} = (s_0 - s)\eta + \langle \boldsymbol{l}_k(\boldsymbol{u}_{\mathrm{L}}), \frac{1}{2}\mathrm{D}^2\boldsymbol{f}(\boldsymbol{u}_{\mathrm{L}})\boldsymbol{r}_k(\boldsymbol{u}_{\mathrm{L}})^2 \rangle \eta^2 + \mathcal{O}(|s|^i |\eta|^j), \quad (i + j \geq 3). \tag{6.23}$$

The claim is that

$$\boldsymbol{l}_k^{\mathrm{T}}(\boldsymbol{u}_{\mathrm{L}})\mathrm{D}^2\boldsymbol{f}(\boldsymbol{u}_{\mathrm{L}})\boldsymbol{r}_k(\boldsymbol{u}_{\mathrm{L}}) = \nabla\lambda_k(\boldsymbol{u}_{\mathrm{L}}).$$

To prove this, first note that

$$\boldsymbol{l}_k(\boldsymbol{u})^{\mathrm{T}}\mathrm{D}\boldsymbol{f}(\boldsymbol{u})\boldsymbol{r}_k(\boldsymbol{u}) = \lambda_k(\boldsymbol{u}), \quad \boldsymbol{u} \in \mathbb{R}^n.$$

Upon differentiating with respect to $\boldsymbol{u}$, evaluating at $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{L}}$, and noting that

$$\begin{aligned} \mathrm{D}\boldsymbol{l}_k^{\mathrm{T}}(\boldsymbol{u}_{\mathrm{L}})\mathrm{D}\boldsymbol{f}(\boldsymbol{u}_{\mathrm{L}})\boldsymbol{r}_k(\boldsymbol{u}_{\mathrm{L}}) + \boldsymbol{l}_k^{\mathrm{T}}(\boldsymbol{u}_{\mathrm{L}})\mathrm{D}\boldsymbol{f}(\boldsymbol{u}_{\mathrm{L}})\mathrm{D}\boldsymbol{r}_k(\boldsymbol{u}_{\mathrm{L}}) &= \lambda_k(\boldsymbol{u}_{\mathrm{L}})(\mathrm{D}\boldsymbol{l}_k^{\mathrm{T}}(\boldsymbol{u}_{\mathrm{L}})\boldsymbol{r}_k(\boldsymbol{u}_{\mathrm{L}}) + \boldsymbol{l}_k^{\mathrm{T}}(\boldsymbol{u}_{\mathrm{L}})\mathrm{D}\boldsymbol{r}_k(\boldsymbol{u}_{\mathrm{L}})) \\ &= \lambda_k(\boldsymbol{u}_{\mathrm{L}})\mathrm{D}_{\boldsymbol{u}}\langle \boldsymbol{l}_k(\boldsymbol{u}), \boldsymbol{r}_k(\boldsymbol{u}) \rangle|_{\boldsymbol{u}=\boldsymbol{u}_{\mathrm{L}}} \\ &= 0, \end{aligned}$$

yields the desired result. Equation (6.23) can now be written as

$$\dot{\eta} = (s_0 - s)\eta + \frac{1}{2}\langle \nabla\lambda_k(\boldsymbol{u}_{\mathrm{L}}), \boldsymbol{r}_k(u_{\mathrm{L}}) \rangle \eta^2 + \mathcal{O}(|s|^i|\eta|^j), \quad (i + j \geq 3).$$

Since the system is genuinely nonlinear, it is not necessary to calculate the terms of $\mathcal{O}(|s|^i|\eta|^j)$ for $i+j \geq 3$. Thus, the bifurcation is of transcritical type. As an application of the Implicit Function Theorem the critical points on W^{c} are given by $\eta = 0\,(\boldsymbol{u} = \boldsymbol{u}_{\mathrm{L}})$ and

$$s = s_0 + \frac{1}{2}\langle \nabla\lambda_k(\boldsymbol{u}_{\mathrm{L}}), \boldsymbol{r}_k(u_{\mathrm{L}}) \rangle \eta + \mathcal{O}(\eta^2).$$

Let $\rho_0 > 0$ be sufficiently small, and assume that $|s - s_0| < \rho_0$. Upon parameterizing the above curve, one has that for each $|\rho| < \rho_0$ there exists an $\eta_{\mathrm{R}} = \eta_{\mathrm{R}}(\rho)$ and $s = s(\rho)$ such that on W^{c}, $\eta = 0$ is connected to η_{R} at $s = s(\rho)$.

Now, in order that the solution approach $\eta = 0$ as $z \to -\infty$, one must necessarily have that $s < s_0$. By construction, one has that

$$\boldsymbol{u}_{\mathrm{R}} = \boldsymbol{u}_{\mathrm{L}} + \eta_{\mathrm{R}} \boldsymbol{r}_k(\boldsymbol{u}_{\mathrm{L}}) + W(\eta_{\mathrm{R}}, s).$$

Upon performing a Taylor expansion for $\lambda_k(\boldsymbol{u})$ at $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{L}}$ and using the above expansion for $\boldsymbol{u}_{\mathrm{R}}$ one sees that

$$\lambda_k(\boldsymbol{u}_{\mathrm{R}}) = \lambda_k(\boldsymbol{u}_{\mathrm{L}}) + \langle \nabla\lambda_k(\boldsymbol{u}_{\mathrm{L}}), \boldsymbol{r}_k(\boldsymbol{u}_{\mathrm{L}})\rangle \eta_{\mathrm{R}} + \mathcal{O}(\eta_{\mathrm{R}}^2).$$

As a consequence, one has that

$$s - \lambda_k(\boldsymbol{u}_{\mathrm{R}}) = -\frac{1}{2}\langle \nabla\lambda_k(\boldsymbol{u}_{\mathrm{L}}), \boldsymbol{r}_k(\boldsymbol{u}_{\mathrm{L}})\rangle \eta_{\mathrm{R}} + \mathcal{O}(\eta_{\mathrm{R}}^2),$$

which, since the system is genuinely nonlinear, implies that $s > \lambda_k(\boldsymbol{u}_{\mathrm{R}})$.

The proof that the second Lax entropy condition follows from the strict hyperbolicity of the system will be left to the interested student. □

6.2. Reduction to planar systems

Now consider equation (6.1) in the case that $\dim(E^{\mathrm{c}}) = 2$. There are then three possible cases to consider:

(a) $\sigma^{\mathrm{c}}(\boldsymbol{A}(0)) = \{0\}$, and the eigenvalue is semi-simple with multiplicity two

(b) $\sigma^{\mathrm{c}}(\boldsymbol{A}(0)) = \{0\}$, and the eigenvalue has geometric multiplicity one and algebraic multiplicity two

(c) $\sigma^{\mathrm{c}}(\boldsymbol{A}(0)) = \{\pm \mathrm{i}\beta\}$, and each eigenvalue is simple.

In each of the cases enumerated above the flow on the center manifold will be described by a planar vector field.

6.2.1. The Hopf bifurcation

Consider equation (6.1) under the condition that $(\boldsymbol{x}_0, 0)$ is a critical point with $\{0\} \not\subset \sigma(\mathrm{D}_x \boldsymbol{f}(\boldsymbol{x}_0, 0))$. As a consequence of the Implicit Function Theorem, for $|\mu| < \mu^*$ there exists a unique curve of critical points $(\boldsymbol{x}(\mu), \mu)$ with $\boldsymbol{x}(0) = \boldsymbol{x}_0$. Suppose that $\mathrm{D}_x \boldsymbol{f}(\boldsymbol{x}(\mu), \mu)$ has the simple eigenvalues $\alpha(\mu) \pm \mathrm{i}\beta(\mu)$ which satisfy

$$\alpha(0) = 0, \quad \alpha'(0) \neq 0, \quad \beta(0) > 0. \tag{6.24}$$

Further suppose that $\sigma^{\mathrm{c}}(\mathrm{D}_x f(x_0, 0)) = \{\pm \mathrm{i}\beta(0)\}$. As discussed in [**18**, Chapter 20.2], the normal form for the equations on W^{c} can then be written as

$$\begin{aligned} \dot{x} &= \alpha(\mu)x - \beta(\mu)y + (a(\mu)x - b(\mu)y)(x^2 + y^2) + \mathcal{O}(|x|^5, |y|^5) \\ \dot{y} &= \beta(\mu)x + \alpha(\mu)y + (b(\mu)x + a(\mu)y)(x^2 + y^2) + \mathcal{O}(|x|^5, |y|^5). \end{aligned} \tag{6.25}$$

In polar coordinates equation (6.25) can be written as

$$\begin{aligned} \dot{r} &= \alpha(\mu)r + a(\mu)r^3 + \mathcal{O}(r^5) \\ \dot{\theta} &= \beta(\mu) + b(\mu)r^2 + \mathcal{O}(r^4). \end{aligned} \tag{6.26}$$

Note that a T-periodic solution to equation (6.25) is equivalent to having a solution $(r(t), \theta(t))$ to equation (6.26) which satisfies

$$r(0) = r(T), \quad \theta(0) = 0, \quad \theta(T) = 2\pi.$$

Upon taking a Taylor expansion and neglecting the higher-order terms in equation (6.26), one finally gets the normal form equations to be studied:

$$\begin{aligned} \dot{r} &= \alpha'(0)\mu r + a(0)r^3 \\ \dot{\theta} &= \beta(0) + \beta'(0)\mu + b(0)r^2. \end{aligned} \tag{6.27}$$

A careful study of equation (6.27) (e.g., see the proof in [**18**, Theorem 20.2.3]) leads to the following result.

Theorem 6.6 (Hopf Bifurcation Theorem)**.** *Consider the system equation (6.1) under the constraints leading to equation (6.24). If $a(0) \neq 0$ and if $|\mu|$ is sufficiently small, then there exists a unique periodic solution of $\mathcal{O}(|\mu|^{1/2})$.*

Remark 6.7. The interested student should consult [**18**, equation (20.2.14)] for an explicit expression for $a(0)$. If $a(0)\alpha'(0) < 0$, then the bifurcation to the periodic orbit is supercritical, whereas if $a(0)\alpha'(0) > 0$, then the bifurcation is subcritical.

6.2.2. The Takens-Bogdanov bifurcation

Now consider equation (6.1) under that assumption that $\boldsymbol{f}(\boldsymbol{0}, \mu) \equiv \boldsymbol{0}$. Furthermore, assume that $\{0\} \subset \boldsymbol{A}(0)$ has geometric multiplicity one and geometric multiplicity two. As discussed in [**18**, Chapter 20.6,Chapter 33.1], the normal form associated with flow on W^{c} is given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu_1 + \mu_2 y + x^2 + bxy, \quad b \in \{-1, +1\}. \end{aligned} \tag{6.28}$$

Many interesting bifurcations occur in equation (6.28); however, we will focus only on two.

Assume that $\mu_1 < 0$. When considering the critical point $(-\sqrt{-\mu_1}, 0)$ the eigenvalues of the linearization are given by

$$\lambda^{\pm} = \frac{1}{2}\left(\mu_2 - \sqrt{-\mu_1} \pm \sqrt{(\mu_2 - \sqrt{-\mu_1})^2 - 8\sqrt{-\mu_1}}\right). \tag{6.29}$$

If one writes $\mu_2 = \sqrt{-\mu_1} + \hat{\mu}\epsilon$ for $0 \le \epsilon \ll 1$, i.e.,

$$\mu_1 = -\mu_2^2 + 2\mu_2\hat{\mu}\epsilon + \mathcal{O}(\epsilon^2), \tag{6.30}$$

then one can rewrite equation (6.29) as

$$\lambda^{\pm} = \frac{1}{2}\hat{\mu}\epsilon \pm \mathrm{i}\sqrt{2}\,(-\mu_1)^{1/4} + \mathcal{O}(\epsilon^2).$$

Thus, upon applying Theorem 6.6 one has that a Hopf bifurcation occurs at $\epsilon = 0$. It can be computed that $a(0) = b/16$ (see [**18**, Chapter 20.6]). Since we are requiring that $\epsilon > 0$, for the bifurcation to occur we must have that $\hat{\mu}b < 0$; hence, it is supercritical. If one assumes that $b = -1$, then the bifurcating solution is stable, whereas if $b = +1$ the bifurcating solution is unstable.

Now let us rescale equation (6.28) in the following manner. For $\epsilon > 0$ introduce the scalings

$$x := \epsilon^2 u, \quad y := \epsilon^3 v, \quad \mu_1 := -\epsilon^4, \quad \mu_2 := \epsilon^2\nu_2, \quad t := \epsilon s,$$

so that equation (6.28) becomes ($' := \mathrm{d}/\mathrm{d}s$)

$$\begin{aligned} u' &= v \\ v' &= -1 + u^2 + \epsilon(\nu_2 v + buv). \end{aligned} \tag{6.31}$$

When $\epsilon = 0$ equation (6.31) is a completely integrable Hamiltonian system with Hamiltonian

$$H(u, v) := \frac{1}{2}v^2 + u - \frac{1}{3}u^3.$$

The system has the homoclinic orbit $(u_0(t), v_0(t))$, where

$$u_0(t) = 1 - 3\,\mathrm{sech}^2(t/\sqrt{2}).$$

Via the use of *Melnikov theory* it can be shown that the homoclinic orbit persists for

$$\nu_2 = \frac{5}{7}b + \mathcal{O}(\epsilon),$$

i.e.,

$$\mu_1 = -\frac{49}{25}\mu_2^2 + \mathcal{O}(\mu_2^{5/2})$$

(compare to equation (6.30)).

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