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EIGENVALUES AND RESONANCES USING THE EVANS FUNCTION

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Abstract. In this expository paper, we discuss the use of the Evans function in finding resonances, which are poles of the analytic continuation of the resolvent. We illustrate the utility of the general theory developed in [13, 14] by applying it to two physically interesting test cases: the linear Schrödinger operator and the linearization associated with the integrable nonlinear Schrödinger equation.

1. Introduction.

Consider the eigenvalue problems

$$H_1 u = \lambda u, \qquad JH_2 u = \lambda u,$$

where each H_j is a self-adjoint operator on a Hilbert space X, and J is an invertible skew-symmetric operator. We assume that each H_j has a finite number of negative eigenvalues and that the continuous spectrum satisfies $\sigma_{\mathrm{ess}}(H_1) = \mathbb{R}^+ \cup \{0\}$ and $\sigma_{\mathrm{ess}}(JH_2) \subset \mathrm{i}\mathbb{R}$. Suppose that each operator H_j depends on a parameter, and that as this parameter is varied eigenvalues are emerging from and/or merging into the continuous spectrum. In this review paper, we address the following questions:

1. From where in the continuous spectrum do eigenvalues arise?
2. At what rate do eigenvalues leave and/or enter the continuous spectrum?
3. How many eigenvalues can leave the continuous spectrum at a given point?

Our goal is to show that the Evans function, which is an analytic tool that counts eigenvalues (see [4, 10, 13, 14, 22] and the references therein), can have something to say about resonances. A resonance can be thought of as eigenvalue in waiting, for even though the associated eigenfunction (anti-bound state) is initially not localized, as the resonance emerges from the essential spectrum and becomes an eigenvalue the eigenfunction becomes localized, i.e., becomes a bound state. As it will be seen below, a resonance is a pole of the analytic continuation of the

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resolvent, which we will capture as roots of the analytic continuation of the Evans function.

In this expository article, these issues will be discussed for two specific examples. The first example, addressed in Section 2, deals with the case when H_1 is a scalar Schrödinger operator. It must be noted that none of the results given in Section 2 is new. Instead, they can, in various and, in fact, much stronger forms, be found in [3, 5, 6, 8, 9, 18, 19, 20, 21, 23, 24, 25, 26] to name a few examples. In these references the eigenvalue problem is studied either via classical scattering theory or via the Fredholm determinant. As it will be seen, the Evans function approach is equivalent to these formulations. The second example discussed in Section 3 is the linearization associated with the nonlinear Schrödinger equation, or some natural generalization thereof. Here we will apply the results presented in [14] to the problem discussed in [13]. The analytic extension is general, and does not require the specific structure of the equation given in Section 3. However, when it comes to the task of evaluating the Evans function, we wish to show how the Inverse Scattering formalism plays a crucial role in the analysis.

2. **The linear Schrödinger operator.** In this section we will construct, and do a perturbation analysis upon, the Evans function for the scalar Schrödinger operator

$$H_1 = -\partial_x^2 + V(x),$$

with $x \in \mathbb{R}$, on the space $X = L^2(\mathbb{R}, \mathbb{C})$. We assume that the real-valued potential decays exponentially with x so that there is a constant C and a rate $a > 0$ such that

$$|V(x)| \leq C\mathrm{e}^{-a|x|}, \qquad x \in \mathbb{R}. \tag{2.1}$$

The theory regarding the Evans function as presented below can be found in its most general form in [13, 7].

The eigenvalue problem associated with the Schrödinger operator H_1 is given by

$$[-\partial_x^2 + V(x)]u = \lambda u, \tag{2.2}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter. Since H_1 is self-adjoint, all of the eigenvalues are real. As a consequence of equation (2.1) the continuous spectrum associated with H_1 is given by

$$\sigma_{\text{ess}}(H_1) = \mathbb{R}^+ \cup \{0\}.$$

Resonances are associated with eigenvalues popping out of the continuum. In order to probe the continuum, it will turn out to be convenient to use the rescaling

$$\lambda = -\gamma^2. \tag{2.3}$$

Equation (2.2) can then be rewritten as

$$[-\partial_x^2 + V(x) + \gamma^2]u = 0. \tag{2.4}$$

As it will be seen below, this definition of the eigenvalue parameter λ allows one to analytically extend the Evans function onto the Riemann surface defined by equation (2.3). This extension is necessary if one wishes to track resonances as they become eigenvalues. The principal sheet of the Riemann surface is given by $\operatorname{Re}\gamma > 0$, and it is here that one finds eigenvalues. The second sheet, $\operatorname{Re}\gamma < 0$, is where one finds resonances of the operator H_1.

We will now construct the Evans function. Upon setting

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} u \\ u_x \end{pmatrix},$$

write the eigenvalue problem (2.4) as the first-order system

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = [M(\gamma) + R(x)]\mathbf{Y}, \tag{2.5}$$

where

$$M(\gamma) = \begin{pmatrix} 0 & 1 \\ \gamma^2 & 0 \end{pmatrix}, \qquad R(x) = \begin{pmatrix} 0 & 0 \\ V(x) & 0 \end{pmatrix}.$$

Note that $\lim_{|x|\to\infty} |R(x)| = 0$ and that the decay is exponentially fast. The eigenvalues of $M(\gamma)$ are given by

$$\mu^{\pm}(\gamma) = \pm\gamma,$$

and the associated eigenvectors are

$$\eta^{\pm}(\gamma) = \begin{pmatrix} 1 \\ \pm\gamma \end{pmatrix}.$$

In particular, the eigenvalues and associated eigenvectors for $M(\gamma)$ are analytic in γ which is the reason for introducing the transformation (2.3). For $\mathrm{Re}\,\gamma > 0$ one can then construct solutions $\mathbf{Y}^{\mathrm{u}}(\gamma, x)$ and $\mathbf{Y}^{\mathrm{s}}(\gamma, x)$ to (2.5) that are analytic in γ for fixed x and that satisfy

$$\lim_{x\to-\infty} \mathbf{Y}^{\mathrm{u}}(\gamma, x)\mathrm{e}^{-\gamma x} = \eta^{+}(\gamma), \qquad \lim_{x\to\infty} \mathbf{Y}^{\mathrm{s}}(\gamma, x)\mathrm{e}^{\gamma x} = \eta^{-}(\gamma).$$

Note that the construction implies that

$$\lim_{x\to-\infty} |\mathbf{Y}^{\mathrm{u}}(\gamma, x)| = 0, \qquad \lim_{x\to\infty} |\mathbf{Y}^{\mathrm{s}}(\gamma, x)| = 0.$$

The Evans function on the Riemann surface is given by

$$E(\gamma) = \det(\mathbf{Y}^{\mathrm{u}}(\gamma, x), \mathbf{Y}^{\mathrm{s}}(\gamma, x)). \tag{2.6}$$

By Abel's formula, the Evans function is in fact independent of x. As a consequence, the solutions $\mathbf{Y}^{\mathrm{u,s}}(\gamma, x)$ comprising the Evans function can be evaluated at any value of x, and in particular can be evaluated in the limit of $x \to +\infty$. This feature will be exploited later in this section.

The importance in the manner in which the Evans function is constructed is seen in the following argument. Suppose that $E(\gamma_0) = 0$ for some $\gamma_0 \in \mathbb{R}^+$, then it is clear that $\mathbf{Y}^{\mathrm{u}}(\gamma_0, x) = \alpha\mathbf{Y}^{\mathrm{s}}(\gamma_0, x)$ for some $\alpha \in \mathbb{C}$. Hence, there is a localized solution to (2.4) when $\gamma = \gamma_0$. As a consequence, γ_0 is an eigenvalue. Similarly, if γ_0 is an eigenvalue with $\mathrm{Re}\,\gamma_0 > 0$, then it is not difficult to convince oneself that $E(\gamma_0) = 0$. The following proposition has then been almost proved.

Proposition 2.1 ([4])**.** *The Evans function is analytic on $\{\gamma \in \mathbb{C};\ \mathrm{Re}\,\gamma > 0\}$. Furthermore, $E(\gamma) = 0$ if, and only if, γ is an eigenvalue, and the order of the root is equal to the algebraic multiplicity of the eigenvalue.*

A different formulation than that presented above is useful when discussing the analytic extension of the Evans function onto the second sheet of the Riemann surface. For $\mathrm{Re}\,\gamma > 0$, the solution $\mathbf{Y}^{\mathrm{u}}(\gamma, x)$ satisfies the integral equation

$$\mathbf{Y}^{\mathrm{u}}(\gamma, x) = \eta^{+}(\gamma)\mathrm{e}^{\gamma x} + \int_{-\infty}^{x} \mathrm{e}^{M(\gamma)(x-y)} R(y)\mathbf{Y}^{\mathrm{u}}(\gamma, y)\,\mathrm{d}y. \tag{2.7}$$

Using the definition of the Evans function given in equation (2.6) and setting $x \gg 1$ yields

$$\begin{aligned} E(\gamma) &= \det(\mathbf{Y}^{\mathrm{u}}(\gamma, x), \mathbf{Y}^{\mathrm{s}}(\gamma, x)) \\ &= \det\left(\mathbf{Y}^{\mathrm{u}}(\gamma, x),\ \eta^{-}(\gamma)\mathrm{e}^{-\gamma x} + \mathrm{o}(\mathrm{e}^{-\gamma x})\right), \end{aligned}$$

so that upon using the formulation in equation (2.7), letting $x \to \infty$, and replacing the integration variable y by x, one gets

$$E(\gamma) = -2\gamma - \int_{-\infty}^{\infty} V(x) Y_1^{\mathrm{u}}(\gamma, x) \mathrm{e}^{-\gamma x} \, \mathrm{d}x. \tag{2.8}$$

Remark 2.2. *For $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$ and $|\gamma| \gg 1$, the system is essentially autonomous, i.e., the influence of the matrix $R(x)$ on the solutions to equation (2.5) becomes negligible (see* [4] *for the details). As a consequence, we have*

$$\lim_{|\gamma|\to\infty} \frac{E(\gamma)}{\gamma} = -2, \qquad \arg(\gamma) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad \operatorname{Re}\gamma > 0.$$

To locate resonances and determine their multiplicity, we need to extend the Evans function across $\operatorname{Re}\gamma = 0$ in an analytic fashion. While the following analysis has been carried out in much more generality in [13, 7], we repeat it here for the sake of completeness and clarity. Thus, set

$$\mathbf{Z}(\gamma, x) = \mathbf{Y}(\gamma, x) \mathrm{e}^{-\gamma x},$$

so that $\mathbf{Z}$ satisfies the ODE

$$\frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}x} = [M(\gamma) - \gamma \, \mathrm{id} + R(x)]\mathbf{Z}. \tag{2.9}$$

We set

$$x = \frac{1}{2\kappa} \ln\left(\frac{1+\tau}{1-\tau}\right), \qquad \text{so that} \qquad \tau = \tanh(\kappa x),$$

and use the relation $x = x(\tau)$ to define $\check{R}(\tau) := R(x(\tau))$. Treating τ as an additional dependent variable, equation (2.9) becomes the autonomous system

$$\begin{aligned} \frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}x} &= [M(\gamma) - \gamma \, \mathrm{id} + \check{R}(\tau)]\mathbf{Z} \\ \frac{\mathrm{d}\tau}{\mathrm{d}x} &= \kappa(1-\tau^2). \end{aligned} \tag{2.10}$$

Solutions of (2.10) with $\tau(x) \in (-1, 1)$ correspond to solutions of (2.9). For such solutions, taking the limits $x \to \pm\infty$ is equivalent to considering $\tau \to \pm 1$. Thus, equation (2.10) is a compactified version of equation (2.9), and we gain some information by allowing τ to vary in $[-1, 1]$ in (2.10). Indeed, the following proposition shows that (2.10) has a well defined flow for $(\mathbf{Z}, \tau)$ in $\mathbb{C}^2 \times [-1, 1]$ and not just on $\mathbb{C}^2 \times (-1, 1)$ where (2.9) and (2.10) are equivalent.

Proposition 2.3 ([4])**.** *If $0 < 2\kappa < a$, then the right-hand side of equation (2.10) is C^1 on $\mathbb{C}^2 \times [-1, 1]$.*

We observe that the plane $\{\tau = -1\}$ is invariant under the flow of (2.10). Furthermore, $\mathbf{Z} = \eta^+(\gamma)$ is an equilibrium in that plane and the linearization about $\mathbf{Z} = \eta^+(\gamma)$ has eigenvalues $0, -2\gamma$ and 2κ. Therefore, as long as

$$-2 \operatorname{Re}\gamma < 2\kappa,$$

i.e., as long as $\operatorname{Re}\gamma > -\kappa$, there is a one-dimensional strong unstable manifold of the equilibrium $(\eta^+(\gamma), -1)$ whose tangent vector points in the τ-direction. As a consequence, there is an $\epsilon \in (0, 1)$ such that this manifold can be written as a function of τ, say $\mathbf{Z}^-(\gamma, \tau)$, for $-1 < \tau < -1+\epsilon$. Furthermore, since the equilibrium in analytic in γ, it follows that $\mathbf{Z}^-(\gamma, \tau)$ is analytic for $-1 < \tau < -1 + \epsilon$. Thus, applying the flow associated with equation (2.10), we see that $\mathbf{Z}^-(\gamma, \tau)$ is well-defined and analytic in γ for $\operatorname{Re}\gamma > -\kappa$ and for $\tau \in [-1, +1)$. In other words, the function $\mathbf{Y}^-(\gamma, x)\mathrm{e}^{-\gamma x}$ is analytic in γ for each x and $\operatorname{Re}\gamma > -\kappa$. Comparing this result with the alternate definition (2.8) of the Evans function gives the following result, the so-called Gap Lemma.

Lemma 2.4. *If* $|V(x)| \leq C\mathrm{e}^{-a|x|}$ *for some* $a > 0$*, then the Evans function is analytic for* $\operatorname{Re}\gamma > -a/2$.

Remark 2.5. *A similar argument shows that* $\mathbf{Y}^{\mathrm{s}}(\gamma, x)\mathrm{e}^{\gamma x}$ *is analytic in* γ *for* $\operatorname{Re}\gamma > -a/2$.

It is worthwhile to remark that the Evans function is analytic for all $\gamma \in \mathbb{C}$ provided the potential $V(x)$ is super-exponential, i.e., provided

$$\lim_{|x|\to\infty} V(x)\mathrm{e}^{N|x|} = 0$$

for every $N > 0$.

Before we consider the meaning of roots of the extended Evans function, we give two complementary interpretations of $E(\gamma)$. First, we connect the Evans function to the transmission coefficient used in Inverse Scattering Theory [1] for Schrödinger operators. Due to the exponential convergence of the coefficient matrix of (2.9), we have

$$\mathbf{Y}^{\mathrm{u}}(\gamma, x)\mathrm{e}^{-\gamma x} = d(\gamma)\eta^+(\gamma) + \mathrm{O}\left(\min(\mathrm{e}^{-2\gamma x}, \mathrm{e}^{-ax})\right)$$

for $x \gg 1$ and $\operatorname{Re}\gamma > 0$, where $d(\gamma)$ is the transmission coefficient. Hence, taking the limit $x \to \infty$ in (2.6), we obtain that

$$d(\gamma) = -\frac{E(\gamma)}{2\gamma}. \tag{2.11}$$

Therefore, roots of the Evans function and the transmission coefficient are in one-to-one correspondence as long as $\gamma \neq 0$. Note, however, that the Evans function is analytic at $\gamma = 0$, while the transmission coefficient $d(\gamma)$ does not necessarily enjoy the same property as evidenced by equation (2.11). At worst, however, $\gamma = 0$ will be a simple pole for the transmission coefficient.

The second interpretation of the Evans function follows the discussion in [25]. Let $G(x, y; \gamma)$ be the Green's function associated with the asymptotic operator $\partial_x^2 - \gamma^2$ and define the analytic Hilbert–Schmidt operator $K(\gamma)$ by

$$K(\gamma)(x, y) = V(x)^{1/2}G(x, y; \gamma)|V(y)|^{1/2},$$

where

$$V(x)^{1/2} = \begin{cases} V(x)/|V(x)|^{1/2} & V(x) \neq 0 \\ 0 & V(x) = 0. \end{cases}$$

The eigenvalue problem (2.4) can then be rewritten as

$$[K(\gamma) + 1]u = 0,$$

so that an eigenvalue exists for $\operatorname{Re}\gamma > 0$ if -1 is an eigenvalue of $K(\gamma)$. The Fredholm determinant is given by

$$d_F(\gamma) = \det(1 + K(\gamma)).$$

The interested reader should consult [23] for a discussion on determinants in the infinite-dimensional case. It is shown in [25] that $d_F(\gamma) = -E(\gamma)/2\gamma$. Hence, from equation (2.11), it is seen that the Fredholm determinant is exactly the transmission coefficient.

Following [3, 5, 6, 8, 9, 18, 19, 20, 21, 23, 24, 25, 26], we can now associate the following terminology with roots of the Evans function. Suppose that $E(\gamma_0) = 0$. If

$\gamma_0 \in \mathbb{R}$ with $\gamma_0 > 0$, then γ_0 is an eigenvalue of H_1, and the associated eigenfunction is a bound state (i.e. exponentially localized). If $\operatorname{Re}\gamma_0 \leq 0$, then γ_0 is called a resonance whose "eigenfunction" is an anti-bound state, i.e., a function that grows exponentially as $|x| \to \infty$.

Next, we address the question of tracking resonances as they become eigenvalues, which is equivalent to tracking roots of the Evans function as they move from the unphysical sheet $\operatorname{Re}\gamma < 0$ of the Riemann surface to the physical sheet $\operatorname{Re}\gamma > 0$. Since H_1 is self-adjoint, a root of the Evans function must pass through $\gamma = 0$ when a resonance becomes an eigenvalue. One interesting issue connected with this transition is to determine the rate at which resonances leave the continuous spectrum through the origin. Suppose that the potential $V(x)$ is replaced by $\alpha V(x)$, so that now the Evans function depends on the spectral parameter γ as well as α. Assume that $E(0,\alpha) = 0$ for $\alpha = \alpha_0$. A Taylor expansion of the Evans function at $(\gamma,\alpha) = (0,\alpha_0)$ yields

$$E(\gamma,\alpha) = \partial_\alpha E(0,\alpha_0)(\alpha-\alpha_0) + \partial_\gamma E(0,\alpha_0)\gamma + \mathrm{O}((\alpha-\alpha_0)^2), \tag{2.12}$$

so that the location of the root γ is given by

$$\gamma = -\frac{\partial_\alpha E(0,\alpha_0)}{\partial_\gamma E(0,\alpha_0)}(\alpha-\alpha_0) + \mathrm{O}((\alpha-\alpha_0)^2)$$

whenever $\partial_\gamma E(0,\alpha_0) \neq 0$. Expansions of this type are given, for example, in [19]. If $\gamma > 0$, so that the zero is on the principal sheet of the Riemann surface, then the perturbed eigenvalue is given by

$$\lambda = -\gamma^2 = \mathrm{O}((\alpha-\alpha_0)^2)$$

which recovers a special case of the result presented in [24].

It is therefore of interest to calculate the relevant terms in the expansion of the Evans function; in particular, $\partial_\gamma E(0,\alpha_0)$. With a slight abuse of notation, we set

$$E(\gamma) := E(\gamma,\alpha_0).$$

Equation (2.8) evaluated at $\gamma = 0$ gives

$$E(0) = -\int_{-\infty}^{\infty} V(x)Y_1^{\mathrm{u}}(0,x)\,\mathrm{d}x = -\lim_{x\to\infty}\partial_x Y_1^{\mathrm{u}}(0,x) \tag{2.13}$$

where the second line follows upon replacing $V(x)Y_1^{\mathrm{u}}(0,x)$ by $\partial_{xx}Y_1^{\mathrm{u}}(0,x)$, which is possible since $Y_1^{\mathrm{u}}(0,x)$ is a solution to (2.4) with $\gamma = 0$, integrating by parts, and using (2.7). Thus, we see that $E(0) = 0$ if, and only if, the solution $Y_1^{\mathrm{u}}(0,x)$ is uniformly bounded. Assume now that indeed $E(0) = 0$. To determine whether the expansion (2.12) is valid, we need to show that the resonance is simple, i.e., that $E'(0) \neq 0$. Following the calculations leading to [12, Lemma 3.5], we can show that

$$E'(0) = -\left[\lim_{x\to-\infty} Y_1^{\mathrm{u}}(0,x) + \lim_{x\to\infty} Y_1^{\mathrm{u}}(0,x)\right] = -\left[1 + \lim_{x\to\infty} Y_1^{\mathrm{u}}(0,x)\right] \tag{2.14}$$

where the second identity follows again from (2.7). Since

$$\lim_{x\to\infty} Y_1^{\mathrm{u}}(0,x) = \lim_{x\to-\infty} Y_1^{\mathrm{u}}(0,x) + \int_{-\infty}^{\infty}\partial_x Y_1^{\mathrm{u}}(0,x)\,\mathrm{d}x = 1 + \int_{-\infty}^{\infty}\partial_x Y_1^{\mathrm{u}}(0,x)\,\mathrm{d}x$$

and since $Y_1^{\mathrm{u}}(0,x)$ satisfies the eigenvalue equation (2.4) with $\gamma = 0$, we can integrate by parts and rewrite equation (2.14) as

$$E'(0) = -2 + \int_{-\infty}^{\infty} xV(x)Y_1^{\mathrm{u}}(0,x)\,\mathrm{d}x. \tag{2.15}$$

For comparison, we remark that $E'(\gamma_0)$ is proportional to

$$\gamma_0 \int_{-\infty}^{\infty} \left[Y_1^{\mathrm{u}}(\gamma_0, x)\right]^2 \,\mathrm{d}x$$

if $\gamma_0 > 0$ is an eigenvalue [11], in which case $Y_1^{\mathrm{u}}(\gamma_0, x)$ is a bound state.

The calculation associated with the expansion (2.12) implicitly assumes that the root at $\gamma = 0$ is simple. It is shown in [25] that $E(0) = 0$ implies $E'(0) \neq 0$ for the half-line problem. As the following simple example shows, this need not be the case for the full-line problem. Set

$$V(x) = \begin{cases} -\beta^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

so that for $x \geq 1$

$$\mathbf{Y}^{\mathrm{u}}(0, x) = [\cos(\beta) + \beta \sin(\beta)] \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \beta \sin(\beta) \begin{pmatrix} x \\ 1 \end{pmatrix}$$

for $x \geq 1$. Using (2.13), it is not difficult to show that

$$E(0) = \beta \sin(\beta)$$

so that $E(0) = 0$ whenever

$$\beta = k\pi, \qquad k \in \mathbb{N}.$$

On the other hand, if $E(0) = 0$, then we see upon using (2.15) or (2.14) that

$$E'(0) = -[1 + \cos(\beta)],$$

so that the root will not be simple if

$$\beta = (2k+1)\pi, \qquad k \in \mathbb{N}.$$

In fact, the Evans function of this example can be constructed explicitly from (2.8) and is given by

$$E(\gamma) = -2\gamma + \beta^2 \int_0^1 \left[\cos\left(\sqrt{\beta^2 - \gamma^2}\, x\right) + \frac{\gamma}{\sqrt{\beta^2 - \gamma^2}} \sin\left(\sqrt{\beta^2 - \gamma^2}\, x\right)\right] \mathrm{e}^{-\gamma x} \,\mathrm{d}x.$$

3. **The linearization JH_2 of the nonlinear Schrödinger equation.** Section 2, and all of the work that we are aware of that uses Fredholm determinants, assumes that the linear operator is self-adjoint. In [13, 14], the concern was with finding resonances for non-self-adjoint operators. In this case, we lose the apriori information that resonances become eigenvalues only through a distinguished point. In particular, if we assume that $\sigma_{\mathrm{ess}}(JH_2) \subset \mathrm{i}\mathbb{R}$, it is at least a possibility that resonances become eigenvalues by crossing at an arbitrary point in the continuous spectrum. As we shall see, resonances can still only cross at certain isolated points in the continuous spectrum. We will use the nonlinear Schrödinger equation hierarchy to illustrate why this is the case and how these points can be identified.

Consider the class of nonlinear integrable Hamiltonian systems given by

$$\mathbf{u}_t = \mathcal{K}(\mathbf{u}), \tag{3.16}$$

where $\mathbf{u} = (r, q) \in L^2(\mathbb{R}, \mathbb{C}^2)$,

$$\mathcal{K}(\mathbf{u}) = -2\sigma_3 \Omega(\mathcal{L}^A(\mathbf{u}))\mathbf{u},$$

σ_3 is the Pauli spin matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\mathcal{L}^A$ is the integro-differential operator

$$\mathcal{L}^A(\mathbf{u})\mathbf{v} = -\frac{\mathrm{i}}{2}\partial_x \mathbf{v} + \mathrm{i}\mathbf{u}\int_{-\infty}^{x} [qv_1 - rv_2]\,\mathrm{d}y,$$

and $\Omega(\cdot) = \mathrm{i}P(\cdot)$, where $P(\cdot)$ is a polynomial with real-valued coefficients. We also use the notation $\mathbf{v} = (v_1, v_2)$. If $P(k) = 1/2 + k^2$, then the evolution equation (3.16) is

$$\begin{aligned} q_t &= \mathrm{i}\left[\frac{1}{2}q_{xx} - q - q^2 r\right] \\ r_t &= -\mathrm{i}\left[\frac{1}{2}r_{xx} - r - qr^2\right], \end{aligned}$$

which becomes the focusing nonlinear Schrödinger equation (NLS) upon setting $r = -q^*$ where q^* denotes the complex conjugate of q. We refer the interested reader to [1, 2] for further details.

We assume that $\mathbf{u}_0$ is a stationary 1-soliton solution of (3.16). We are interested in the spectrum of the linearization $\mathcal{K}'(\mathbf{u}_0)$ about $\mathbf{u}_0$ and, in particular, in eigenvalues λ for which

$$\mathcal{K}'(\mathbf{u}_0)\mathbf{u} = \lambda\mathbf{u}$$

has a non-zero solution $\mathbf{u}$ in $L^2(\mathbb{R}, \mathbb{C}^2)$. The eigenvalue problem for the focusing NLS reads

$$\begin{aligned} \lambda q &= \mathrm{i}\left[\frac{1}{2}q_{xx} - q - 2q_0 r_0 q - q_0^2 r\right] \\ \lambda r &= -\mathrm{i}\left[\frac{1}{2}r_{xx} - r - 2q_0 r_0 r - r_0^2 q\right], \end{aligned} \tag{3.17}$$

where $\mathbf{u}_0 = (q_0, r_0)$.

First, the essential spectrum of $\mathcal{K}'(\mathbf{u}_0)$ is given by

$$\{\lambda \in \mathbb{C};\ \lambda = \pm 2\Omega(k),\ k \in \mathbb{R}\} \subset \mathrm{i}\mathbb{R}.$$

Also, it was shown in [14] that $\lambda = 0$ is the only eigenvalue of $\mathcal{K}'(\mathbf{u}_0)$. These eigenvalues at $\lambda = 0$ are due to the invariances associated with equation (3.16). We will recover this result later in this section for the polynomial dispersion relation P that gives the nonlinear Schrödinger equation. The question that we wish to investigate is the location of the spectrum upon adding perturbations to (3.16). In particular, we are interested in resonances and eigenvalues that emerge from the essential spectrum. This problem can be solved if we can construct, and compute, the Evans function associated with the operator $\mathcal{K}'(\mathbf{u}_0)$.

The key to calculating the Evans function is to exploit Inverse Scattering Theory which is possible since (3.16) is integrable. Indeed, the underlying linear scattering problem associated with the nonlinear operator $\mathcal{K}(\mathbf{u})$ is the Zakharov-Shabat problem [2]

$$\mathbf{v}_x = \begin{pmatrix} -\mathrm{i}k & q_0(x) \\ -q_0^*(x) & \mathrm{i}k \end{pmatrix} \mathbf{v}$$

where $k \in \mathbb{C}$ is a complex parameter. The Jost functions are solutions to the Zakharov-Shabat eigenvalue problem that satisfy certain boundary conditions at $x = \pm\infty$. Appropriate quadratic combinations of the Jost functions define the adjoint squared eigenfunctions which we denote[1] by $\Psi^{\mathrm{A}}(k,x)$, defined for $\operatorname{Im} k \geq 0$, and $\bar{\Psi}^{\mathrm{A}}(k,x)$, defined for $\operatorname{Im} k \leq 0$. The adjoint squared eigenfunctions are crucial ingredients when applying Soliton Perturbation Theory [16, 17]. As we shall see below, they can also be used to explicitly calculate the Evans function associated with $\mathcal{K}'(\mathbf{u}_0)$. For $k \in \mathbb{R}$, the adjoint squared eigenfunctions satisfy the identities

$$[\mathcal{L}^{\mathrm{A}}(\mathbf{u}) - k]\Psi^{\mathrm{A}}(k,x) = [\mathcal{L}^{\mathrm{A}}(\mathbf{u}) - k]\bar{\Psi}^{\mathrm{A}}(k,x) = 0.$$

Furthermore, they have the property that, for fixed x, $\Psi^{\mathrm{A}}(k,x)$ is analytic in k for $\operatorname{Im} k > 0$, while $\bar{\Psi}^{\mathrm{A}}(k,x)$ is analytic in k for $\operatorname{Im} k < 0$. In addition, they have the asymptotics

$$\lim_{x\to-\infty} \Psi^{\mathrm{A}}(k,x)\mathrm{e}^{2\mathrm{i}kx} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \qquad \lim_{x\to\infty} \Psi^{\mathrm{A}}(k,x)\mathrm{e}^{2\mathrm{i}kx} = a(k)^2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{3.18}$$

$$\lim_{x\to-\infty} \bar{\Psi}^{\mathrm{A}}(k,x)\mathrm{e}^{-2\mathrm{i}kx} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \lim_{x\to\infty} \bar{\Psi}^{\mathrm{A}}(k,x)\mathrm{e}^{-2\mathrm{i}kx} = \bar{a}(k)^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.19}$$

The functions $a(k)$ and $\bar{a}(k)$ are the transmission coefficients for the Zakharov-Shabat eigenvalue problem. They are analytic for $\operatorname{Im} k > 0$ and $\operatorname{Im} k < 0$, respectively. It is known, see [14] for references, that the adjoint squared eigenfunctions and the transmission coefficients can be extended analytically across the line $\operatorname{Im} k = 0$. The statement that the wave $\mathbf{u}_0$ is a 1-soliton implies in particular that both $a(k)$ and $\bar{a}(k)$ have a unique root for $\operatorname{Im} k \neq 0$, that this root is in fact simple, and that both $a(k)$ and $\bar{a}(k)$ are non-zero for $k \in \mathbb{R}$.

Lemma 3.1 ([14])**.** *The adjoint squared eigenfunctions satisfy*

$$\begin{aligned} [\mathcal{K}'(\mathbf{u}_0) - 2\Omega(k)]\sigma_3\Psi^{\mathrm{A}}(k,x) &= 0 \qquad && \operatorname{Im} k \geq 0 \\ [\mathcal{K}'(\mathbf{u}_0) + 2\Omega(\bar{k})]\sigma_3\bar{\Psi}^{\mathrm{A}}(\bar{k},x) &= 0 \qquad && \operatorname{Im} \bar{k} \leq 0. \end{aligned}$$

We are now in the position to calculate the Evans function. For the sake of clarity, we consider first the specific dispersion relation

$$\Omega(\ell) = \mathrm{i}\left(\frac{1}{2} + \ell^2\right) \tag{3.20}$$

of the nonlinear Schrödinger equation. For this case an explicit expression is known [17] for the adjoint squared eigenfunctions given in Lemma 3.1; however, as it will be seen below, these expressions are not necessary in order to do a calculation. The results for more general dispersion relations will be discussed later.

Thus, consider (3.20) and note that the associated continuous spectrum consists of the elements $\lambda = \pm\mathrm{i}(1 + 2\ell^2)$ where ℓ varies in $\mathbb{R}$, i.e., it is the imaginary axis minus the interval $(-\mathrm{i}, \mathrm{i})$. The transmission coefficients of the 1-soliton can be chosen [15, 17] so that both are non-zero everywhere on their domain of definition except for

$$a(\mathrm{i}) = \bar{a}(-\mathrm{i}) = 0, \qquad a'(\mathrm{i}) \neq 0, \qquad \bar{a}'(-\mathrm{i}) \neq 0. \tag{3.21}$$

We focus on elements of the essential spectrum with positive imaginary part. Note that $\ell = 0$ corresponds to a double root of the relation $\lambda(\ell) = \pm\mathrm{i}(1 + 2\ell^2)$. To define the Evans function near $\lambda = \mathrm{i}$, it is therefore convenient to use an appropriate Riemann surface as in Section 2. The Riemann surface in this problem is defined by

$$\gamma^2 = 2(\lambda - \mathrm{i}), \tag{3.22}$$

[1] Notation: We denote by q^* the complex conjugate of a complex number q. Thus, $\bar{q}$ does *not* refer to the complex conjugate.

so that $\lambda = \mathrm{i} + \gamma^2/2$. The primary sheet of the Riemann surface is given by

$$\arg(\gamma) \in \left(-\frac{3\pi}{4}, \frac{\pi}{4}\right],$$

with $\operatorname{Re}\lambda \geq 0$ corresponding to $\arg(\gamma) \in [-\pi/4, \pi/4]$. Since we wish to exploit Lemma 3.1, we choose, for each λ near $\lambda_{\mathrm{bp}} = \mathrm{i}$, numbers k near 0 and $\bar{k}$ near $-\mathrm{i}$ such that

$$\lambda = 2\Omega(k) = \mathrm{i}(1 + 2k^2), \qquad \lambda = -2\Omega(\bar{k}) = -\mathrm{i}(1 + 2\bar{k}^2),$$

and so that we have $\operatorname{Im} k > 0$ on the principal sheet of the Riemann surface and $\operatorname{Im} \bar{k} < 0$ for each γ near 0. Using the transformation (3.22) and Lemma 3.1, we see that, for all γ near zero,

$$\begin{aligned} &\sigma_3 \Psi^{\mathrm{A}}(k, x), \qquad \text{with} \quad k(\gamma) = -\frac{\gamma}{2}\mathrm{e}^{-\mathrm{i}\pi/4} \\ &\sigma_3 \bar{\Psi}^{\mathrm{A}}(\bar{k}, x), \qquad \text{with} \quad \bar{k}(\gamma) = -\frac{1}{2}\sqrt{-4 + \mathrm{i}\gamma^2} \quad \text{and} \quad \sqrt{-1} = \mathrm{i} \end{aligned} \tag{3.23}$$

are two solutions to the eigenvalue problem (3.17). Both of these solutions decay exponentially fast to zero as $x \to -\infty$ on the principal sheet of the Riemann surface. Furthermore, as a consequence of (3.18) and (3.19), we know their asymptotics as $x \to \infty$. From now on, we regard k and $\bar{k}$ as functions of γ defined via (3.23).

Next, we rewrite the eigenvalue problem (3.17) as an ODE

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = [M(\gamma) + R(x)]\mathbf{Y}, \qquad \mathbf{Y} = \begin{pmatrix} q \\ r \\ q_x \\ r_x \end{pmatrix},$$

where γ is given via (3.22), $\mathbf{Y} \in \mathbb{C}^4$, and

$$M(\gamma) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2(2 - \mathrm{i}\gamma^2) & 0 & 0 & 0 \\ 0 & 2\mathrm{i}\gamma^2 & 0 & 0 \end{pmatrix}, \qquad R(x) = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2q_0 r_0 & q_0^2 & 0 & 0 \\ r_0^2 & 2q_0 r_0 & 0 & 0 \end{pmatrix}.$$

Note that $|R(x)| \to 0$ as $|x| \to \infty$. The transformation (3.22) ensures that the eigenvalues and associated eigenvectors of $M(\gamma)$ are analytic. On the principal sheet of the Riemann surface, we can define two solutions[2]

$$\mathbf{Y}_1^{\mathrm{u}}(\gamma, x) = \begin{pmatrix} 1 \\ \partial_x \end{pmatrix} \sigma_3 \Psi^{\mathrm{A}}(k(\gamma), x), \qquad \mathbf{Y}_2^{\mathrm{u}}(\gamma, x) = \begin{pmatrix} 1 \\ \partial_x \end{pmatrix} \sigma_3 \bar{\Psi}^{\mathrm{A}}(\bar{k}(\gamma), x)$$

that decay exponentially to zero as $x \to -\infty$, where k and $\bar{k}$ are defined in (3.23). Since they are given in terms of the squared eigenfunctions, we can extend the above two solutions analytically onto the second sheet of the Riemann surface. Analogously, there are two other solutions, analytic in γ for fixed x, that decay

[2]Note that the subscripts of $\mathbf{Y}_j^{\mathrm{u}}$ do *not* denote the components of the vectors.

exponentially as $x \to \infty$ for γ on the principal sheet of the Riemann surface via

$$\lim_{x\to\infty} \mathbf{Y}_1^{\mathrm{s}}(\gamma,x)\mathrm{e}^{-2\mathrm{i}k(\gamma)x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\gamma\mathrm{e}^{\mathrm{i}\pi/4} \end{pmatrix}, \tag{3.24}$$

$$\lim_{x\to\infty} \mathbf{Y}_2^{\mathrm{s}}(\gamma,x)\mathrm{e}^{2\mathrm{i}\bar{k}(\gamma)x} = \begin{pmatrix} 1 \\ 0 \\ \mathrm{i}\sqrt{-4+\mathrm{i}\gamma^2} \\ 0 \end{pmatrix}.$$

These solutions can also be extended analytically onto the second sheet of the Riemann surface. The Evans function is then defined as

$$E(\gamma) = \det\left(\mathbf{Y}_1^{\mathrm{u}}, \mathbf{Y}_2^{\mathrm{u}}, \mathbf{Y}_1^{\mathrm{s}}, \mathbf{Y}_2^{\mathrm{s}}\right)(\gamma, x).$$

By Abel's formula, it is independent of x since $\mathrm{tr}[M(\gamma)+R(x)] = 0$. The Evans function has the property that it vanishes at γ on the principal sheet if, and only if, the associated λ is an eigenvalue. Furthermore, the order of the root γ is the algebraic multiplicity of the associated eigenvalue λ.

Finally, we evaluate the Evans function. As a consequence of (3.18) and (3.19), we have the asymptotics

$$\lim_{x\to\infty} \mathbf{Y}_1^{\mathrm{u}}(\gamma,x)\mathrm{e}^{2\mathrm{i}k(\gamma)x} = a(k(\gamma))^2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \gamma\mathrm{e}^{\mathrm{i}\pi/4} \end{pmatrix} \tag{3.25}$$

$$\lim_{x\to\infty} \mathbf{Y}_2^{\mathrm{u}}(\gamma,x)\mathrm{e}^{-2\mathrm{i}\bar{k}(\gamma)x} = \bar{a}(\bar{k}(\gamma))^2 \begin{pmatrix} 1 \\ 0 \\ -\mathrm{i}\sqrt{-4+\mathrm{i}\gamma^2} \\ 0 \end{pmatrix}.$$

Taking the limit $x \to \infty$ and using the asymptotics given in equations (3.24)-(3.25), we obtain

$$\begin{aligned} E(\gamma) &= 4\mathrm{e}^{-\mathrm{i}\pi/4} a(k)^2 \bar{a}(\bar{k})^2 \gamma \sqrt{-4+\mathrm{i}\gamma^2} \\ &= 8a(k)^2\bar{a}(\bar{k})^2\sqrt{\lambda-\mathrm{i}}\sqrt{\lambda+\mathrm{i}}, \end{aligned}$$

where k and $\bar{k}$ depend on γ via (3.23). The second line follows from the definition of γ given in (3.22). Note that, as in Section 2, the transmission coefficients play a crucial role in the calculation. We also remark that the choice of transmission coefficients made in (3.21) together with the definition of k and $\bar{k}$ imply that the Evans function is non-zero except at the branch points $\lambda = \pm\mathrm{i}$ and at $\lambda = 0$ which is a root of order four. Finally, note that

$$\lim_{\gamma\to 0} \frac{E(\gamma)}{\gamma} = C$$

for some non-zero constant C. Thus, the resonances are located exactly at the branch points of the continuous spectrum which sit at the edges of the spectrum. Since the Evans function is smooth on the Riemann surface, at most one eigenvalue will emerge from the edges $\lambda = \pm\mathrm{i}$ of the essential spectrum upon perturbing the underlying PDE. As in Section 2, if the perturbation is of $\mathrm{O}(\epsilon)$, then the perturbed eigenvalue will be $\mathrm{O}(\epsilon^2)$ away from the edge of the continuum.

Lastly, we comment on the more general dispersion relation

$$\Omega(k) = \frac{\mathrm{i}(1 + 2k^{2j})}{2}, \qquad j \in \mathbb{N},$$

where the nonlinear Schrödinger equation is recovered with $j = 1$. The Riemann surface is defined by

$$\gamma^{2j} = 2(\lambda - \mathrm{i}),$$

so that $\lambda = \mathrm{i} + \gamma^{2j}/2$. This time, we have

$$\lim_{\gamma \to 0} \frac{E(\gamma)}{\gamma^{j^2}} = C$$

for some non-zero constant C [14]. Again, as shown in [14], there are no other resonance points. Thus, as in the case of $j = 1$, resonances can become eigenvalues only by leaving through the edge of the continuous spectrum. For general polynomial dispersion relationships, the resonance points are those points $\lambda = 2\Omega(k)$ where $\Omega'(k) = 0$ [14]. Hence, it is possible for eigenvalues to emerge from inside the continuous spectrum and not just the edge. We refer to [14] for further details.

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