Stability analysis of pulses via the Evans function: dissipative systems

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Abstract. The linear stability analysis of pulses is considered in this review article. It is shown herein the general manner in which the Evans function, an analytic tool whose zeros correspond to eigenvalues, is constructed. Furthermore, the construction is done explicitly for the linearization of the nonlinear Schrödinger equation about the 1-soliton solution. In another explicit calculation, it is shown how the Evans function can be used to track the nonzero eigenvalues arising from a dissipative perturbation of the nonlinear Schrödinger equation which arises in the context of pulse propagation in nonlinear optical fibers.

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1. Introduction

One of the more exciting areas in applied mathematics is the study of the dynamics associated with the propagation of information. Phenomena of interest include the transmission of impulses in nerve fibers, the transmission of light down an optical fiber, and phase transitions in materials. The nature of the system dictates that the relevant and important effects occur along one axial direction. The models formulated in these areas exhibit many other effects, but it is these nonlinear waves that are the raison-d'être of the models. The demands on the mathematician for techniques to analyze these models may best be served by developing methods tailored to determining the local behavior of solutions near these structures. The most basic question along these lines is the stability of the waves relative to perturbations in the initial data. Only waves that are stable can be reasonably expected to be physically realizable. By the same token, the presence of any instability and understanding its source can be crucial if the goal is to control the wave to a stable configuration.

There have been some striking advances over recent years in the development of stability techniques for nonlinear waves such as fronts, pulses and wave-trains. A motivating force behind these stability methods is the desire for "rules-of-thumb" or "principles" that can be adopted by physical scientists. An example of such a rule would be the result that a travelling wave solution of a (scalar) reaction-diffusion equation is stable exactly when it is monotone (see Fife and McLeod [16]). In other words: fronts are stable, but pulses are not. This is a simple consequence of classical Sturm-Liouville theory applied to the eigenvalue equations of the equations linearized at the wave, and yields a relationship between the structure of the wave itself and its stability properties. Armed with this information, one has a "rule-of-thumb" for how to discriminate between stable and unstable waves in this system. Another example is the Vakhitov-Kolokov criterion, which is applicable to systems such as the nonlinear Schrödinger equation. Letting $P(\omega)$ represent the power of the underlying wave, the wave is stable if $\mathrm{d}P/\mathrm{d}\omega > 0$, and unstable otherwise. That such rules should exist in more general systems and for nonlinear waves with more complicated structure has driven much of the research in this area.

The key information for stability is contained in the linearization of the PDE about the wave. In many cases, location of the spectrum suffices to determine the stability, i.e., spectrum in the left half plane corresponds to stable directions and that in the right half plane corresponds to unstable directions. In dissipative systems this basic linear information is definitive. However, there are a number of interesting problems in which more subtle information about the linearized system is sought; these tend to occur in

conservative or near-conservative systems for which there is considerable spectrum either on or touching the imaginary axis.

One tool in particular has come to stand out as central in stability investigations of nonlinear waves. The Evans function is an analytic function whose zeroes give the eigenvalues of the linearized operator, with the order of the zero and the multiplicity of the eigenvalue matching. The Evans function is a generalization to systems of PDEs of the transmission coefficient from quantum mechanics and was first formulated by Evans for a specific class of systems, see [**12**–**15**]. Evans was interested in the stability of nerve impulses and formulated a category of equations that he named "nerve impulse equations". This class of equations had a special property that made the formulation of this function straightforward. In his paper, he used the notation $D(\lambda)$ to connote "determinant", as it played the same role as the determinant of an eigenvalue matrix in finite-dimensional problems. Jones [**18**] used Evans' idea to solve the problem for the stability of the travelling pulse (nerve impulse) of the FitzHugh-Nagumo system. It was named by Jones as the Evans function and the notation $E(\lambda)$ is now in common usage. The first general definition of the Evans function was given by Alexander et al. [**3**]. Although based on Evans' idea, these authors put it on a new conceptual plane in order to give a clear, general definition.

We will now focus on a representative problem, which motivates much of the discussion in this review article. Pulse propagation in a standard single-mode optical fiber is modelled by an equation of the form

$$\mathrm{i}q_t + \frac{1}{2}q_{xx} + |q|^2 q = \epsilon R(x, t, q, q^*). \tag{1.1}$$

Here q represents the slowly-varying envelope of the rapidly varying wave, the term $|q|^2 q$ represents the nonlinear response of the fiber, and the perturbation term R incorporates additional (nonlinear) effects such as Raman scattering, phase amplification, spectral filtering, and impurities in the fiber. A pulse solution corresponds to a "bit" of information propagating down the optical fiber, and is realized as a homoclinic orbit for the underlying travelling wave ODE. Equation (1.1), in addition to being a physically realistic model, is amenable to an extensive and thorough analysis; furthermore, it incorporates many of the mathematical issues and difficulties present in more complex perturbed Hamiltonian systems.

The structure and stability of the pulse is well-known for the unperturbed problem, as equation (1.1) is an integrable system (see Section 4). The pulse can be well-described by four parameters: amplitude, wave speed, position, and phase. From this it can be seen that the origin of the complex plane is an eigenvalue with geometric multiplicity (g.m.) two and algebraic multiplicity (a.m.) four. The fact that a.m.$\neq$g.m. is a reflection of the fact that the system is Hamiltonian. Furthermore, the only other spectrum is essential, and it resides completely on the imaginary axis.

There are several fundamental stability issues that arise for the perturbed problem, assuming that some subset of the family of pulse solutions is chosen by the perturbation. The first is the fate of the spectrum of the origin. If the perturbation is dissipative but breaks neither of the symmetries, then the small bifurcating eigenvalues will typically be of $\mathrm{O}(\epsilon)$. How does one systematically and generally capture the location of these small eigenvalues? This problem can be thought of as lying in the realm of classical bifurcation theory, and is addressed in Section 5.

A more subtle effect to be understood is the influence of the perturbation on the essential spectrum. The location of the essential spectrum after the perturbation is well-understood; however, what is less understood is determining the location of point eigenvalues in or near near the essential spectrum which popped out after the perturbation. In the physics literature these eigenvalues are often called internal modes, as for perturbed Hamiltonian systems they often reside on the imaginary axis. An understanding of this phenomena is important, as any or all of these eigenvalues can lead to an instability. This creation of internal modes has been termed an Edge Bifurcation [**24**]. This naming is due to the fact that for equation (1.1) the bifurcation occurs only at the edge of the essential spectrum. The questions to be answered in this problem are: (a) from which points in the essential spectrum can an edge bifurcation occur? (b) how many eigenvalues will arise from a particular bifurcation? This problem can be thought of as lying outside classical bifurcation theory, primarily because it is not clear as to the location of the bifurcation point in the spectral plane. This issue is briefly addressed in Section 4.1, and much more extensively in [**26**].

This article is outlined in the following way. In Section 2 we show how to construct the Evans function for a simple example. In Section 3 we give several equivalent constructions of the Evans function, and discuss its properties. In Section 4 we use the Evans function to compute the spectrum for equation (1.1)

in the unperturbed case. Finally, in Section 5 we compute the spectrum for equation (1.1) for a particular dissipative perturbation.

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2. Basic example

Consider the scalar reaction-diffusion equation

$$u_t = u_{xx} - u + 2u^3, \tag{2.1}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. It is easy to check that equation (2.1) has a pulse solution which is given by $u(x) = U(x)$, where $U(x) = \operatorname{sech}(x)$. Linearizing equation (2.1) yields the linear eigenvalue problem

$$p'' - (1 - 6U^2(x))p = \lambda p, \quad ' = \frac{\mathrm{d}}{\mathrm{d}x}. \tag{2.2}$$

Since equation (2.1) is translation invariant, $\lambda = 0$ is an eigenvalue with associated eigenfunction $U'(x)$. Since $U(x)$ is a nonmonotone solution, as an easy application of Sturm-Liouville theory one can then deduce that there is one positive eigenvalue, and hence the wave is unstable. The purpose of this section is to show how the Evans function can be used to deduce the same conclusion. The idea is that by examining a relatively simple problem, the reader will then have a "blueprint" for the theory and ideas that lie ahead.

Upon setting $\mathbf{Y} = (p,q)^{\mathrm{T}}$, write the eigenvalue problem equation (2.2) as the first-order system

$$\mathbf{Y}' = (M(\lambda) + R(x))\mathbf{Y}, \tag{2.3}$$

where

$$M(\lambda) = \begin{pmatrix} 0 & 1 \\ 1+\lambda & 0 \end{pmatrix}, \quad R(x) = \begin{pmatrix} 0 & 0 \\ -6U^2(x) & 0 \end{pmatrix}.$$

It is important to note that $\lim_{|x|\to+\infty} |R(x)| = 0$, and that the decay is exponentially fast. For the rest of this discussion it will be assumed that $\operatorname{Re}\lambda > -1$. The eigenvalues of $M(\lambda)$ are given by

$$\mu^{\pm}(\lambda) = \pm\sqrt{1+\lambda},$$

and the associated eigenvectors are

$$\eta^{\pm}(\lambda) = (1, \mu^{\pm}(\lambda))^T.$$

One can construct solutions $\mathbf{Y}^{\pm}(\lambda,x)$ to equation (2.3) which satisfy

$$\lim_{x\to\pm\infty} \mathbf{Y}^{\pm}(\lambda,x)\mathrm{e}^{-\mu^{\mp}(\lambda)x} = \eta^{\mp}(\lambda);$$

note that the construction implies that $\lim_{x\to\pm\infty} |\mathbf{Y}^{\pm}(\lambda,x)| = 0$. The Evans function is given by

$$E(\lambda) = \det(\mathbf{Y}^-, \mathbf{Y}^+)(\lambda,x), \tag{2.4}$$

and by Abel's formula it is independent of x.

The importance in the manner in which the Evans function is constructed is seen in the following argument. Suppose that $E(\lambda_0) = 0$ for some λ_0 with $\operatorname{Re}\lambda_0 > -1$. It is then clear that $\mathbf{Y}^-(\lambda_0,x) = \alpha\mathbf{Y}^+(\lambda_0,x)$ for some $\alpha \in \mathbb{C}$. Hence, there is a localized solution to equation (2.2) when $\lambda = \lambda_0$, so that λ_0 is an eigenvalue. Similarly, if λ_0 is an eigenvalue with $\operatorname{Re}\lambda_0 > -1$, then it is not difficult to convince oneself that $E(\lambda_0) = 0$. The following proposition has then been almost proved.

Proposition 2.1. *Set $\Omega = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -1\}$. The Evans function is analytic on Ω. Furthermore, $E(\lambda) = 0$ if an only if λ is an eigenvalue, and the order of the zero is equal to the algebraic multiplicity of the eigenvalue.*

It is of interest to relate the Evans function to the transmission coefficient associated with the Inverse Scattering Transform [1, 2]. For $\operatorname{Re}\lambda > -1$ and $x \gg 1$ the solution $\mathbf{Y}^-(\lambda, x)$ has the asymptotics

$$\mathbf{Y}^-(\lambda, x) = a(\lambda)\eta^+(\lambda)\mathrm{e}^{\mu^+(\lambda)x} + b(\lambda)\eta^-(\lambda)\mathrm{e}^{\mu^-(\lambda)x} + \mathrm{O}(x^{-1}). \tag{2.5}$$

Here $a(\lambda)$ is the transmission coefficient, and $a(\lambda_0) = 0$ if and only if λ_0 is an eigenvalue. Using the definition of the Evans function given in equation (2.4) and letting $x \to +\infty$ yields

$$a(\lambda) = -\frac{E(\lambda)}{2\sqrt{1+\lambda}}; \tag{2.6}$$

hence, for $\operatorname{Re}\lambda > -1$ one has that $E(\lambda) = 0$ if and only if $a(\lambda) = 0$.

We will now use the Evans function to show that the wave is unstable. For $\lambda \in \Omega$ with $|\lambda| \gg 1$, the system is essentially autonomous, i.e., the influence of the matrix $R(x)$ on the solutions to equation (2.3) becomes negligible (see [3] for the details). This is equivalent to the transmission coefficient being unity for large λ. As a consequence, it is easy to see from equation (2.6) that for $\lambda \in \Omega$,

$$\lim_{|\lambda|\to+\infty} \frac{E(\lambda)}{\sqrt{1+\lambda}} = -2. \tag{2.7}$$

Since $\lambda = 0$ is an eigenvalue, one has that $E(0) = 0$. As a consequence of equation (2.7) it is seen that the Evans function is negative for large real positive λ; therefore, the wave will necessarily be unstable if $E'(0) > 0$. We now proceed to make that calculation.

By construction one has that $\mathbf{Y}^-(0, x) = \mathbf{Y}^+(0, x) = (U'(x), U''(x))^{\mathrm{T}}$. Taking a derivative with respect to λ yields

$$E'(0) = \det(\partial_\lambda(\mathbf{Y}^- - \mathbf{Y}^+),\, \mathbf{Y}^+)(0, x). \tag{2.8}$$

Now, upon observation of equation (2.3) it is easy to see that at $\lambda = 0$,

$$(\partial_\lambda \mathbf{Y}^\pm)' = (M(0) + R(x))\partial_\lambda \mathbf{Y}^\pm + M'(0)\mathbf{Y}^\pm. \tag{2.9}$$

Note that

$$M'(0)\mathbf{Y}^\pm = \begin{pmatrix} 0 \\ U'(x) \end{pmatrix}.$$

Upon solving equation (2.9) via variation of parameters, one finds that

$$\partial_\lambda(\mathbf{Y}^- - \mathbf{Y}^+)(0, x) = \left(-\int_{-\infty}^{+\infty} (U'(x))^2\,\mathrm{d}x\right)\mathbf{u}_2(x) + C_1\mathbf{u}_1(x) \tag{2.10}$$

for some constant C_1. Here $\mathbf{u}_1(x) = \mathbf{Y}^-(0, x)$, and $\mathbf{u}_2(x)$ is another solution to equation (2.3) at $\lambda = 0$ such that $\det(\mathbf{u}_1,\, \mathbf{u}_2)(x) = 1$. Substitution of the result of equation (2.10) into the expression of equation (2.8) then yields that

$$\begin{aligned} E'(0) &= \left(-\int_{-\infty}^{+\infty} (U'(x))^2\,\mathrm{d}x\right)\det(\mathbf{u}_2,\, \mathbf{u}_1)(x) \\ &= \int_{-\infty}^{+\infty} (U'(x))^2\,\mathrm{d}x. \end{aligned} \tag{2.11}$$

Thus, one has the existence of an odd number of real positive zeros. As previously mentioned, there is exactly one.

While we will not discuss the issue in any more detail here, this idea of proving the instability of a wave has proven to be very fruitful. The interested reader should consult, for example, [4–6, 9, 10, 17, 19] and the references therein.

2.1. Alternative definition

Recall the definition of the Evans function given in equation (2.4). There is an equivalent definition which was first used by Evans [12, 13, 14, 15], and which was later used by Jones [18] to determine the stability of the fast travelling pulse to the Fitzhugh-Nagumo equation (also see [33, 34]). Consider the adjoint equation to equation (2.3) given by

$$\mathbf{Z}' = -(M(\lambda)^* + R(x))^{\mathrm{T}}\mathbf{Z}. \tag{2.12}$$

As in the construction of the solutions $\mathbf{Y}^{\pm}(\lambda, x)$ to equation (2.3), one can construct a solution $\mathbf{Z}^{+}(\lambda, x)$, analytic in λ for fixed x, to equation (2.12) which satisfies

$$\lim_{x\to+\infty} \mathbf{Z}^{+}(\lambda, x)\mathrm{e}^{-\mu^{-}(\lambda)^* x} = (\mu^{-}(\lambda)^*, 1)^{\mathrm{T}}.$$

The Evans function can be written as

$$E(\lambda) = \langle \mathbf{Y}^{-}(\lambda, x), \mathbf{Z}^{+}(\lambda, x)\rangle, \tag{2.13}$$

where $\langle\cdot,\cdot\rangle$ represents the standard inner product on $\mathbb{C}^2$. A generalization of this formulation will be given in the subsequent section.

3. Construction of the Evans function

Now that we have an idea as to how the Evans function is constructed for scalar reaction-diffusion equations, let us consider more general PDEs of the class

$$u_t = Bu_{xx} + f(u, u_x),$$

where it is assumed that the initial value problem is well-posed. Upon setting $z = x - ct$, the PDE becomes

$$u_t = Bu_{zz} + cu_z + f(u, u_z).$$

For the PDE at hand it will be assumed that there is a travelling pulse $U(z)$ and a constant state U^* such that $|U(z) - U^*| \to 0$ exponentially fast as $|z| \to \infty$. If one desires, this restriction can be relaxed to consider travelling fronts which connect a constant state U_- to a state U_+; however, upon doing so one only increases the notational complexity without increasing the generality of all that follows (see [3] for further details).

After linearizing the PDE about the travelling wave, the eigenvalue problem can be rewritten as the first-order system

$$\mathbf{Y}' = (M(\lambda) + R(z))\mathbf{Y}, \quad ' = \mathrm{d}/\mathrm{d}z, \tag{3.1}$$

where $\lambda \in \mathbb{C}$ is the eigenvalue parameter and the $n \times n$ matrix $|R(z)| \to 0$ exponentially fast as $|z| \to \infty$. If λ is not in the continuous spectrum, then the matrix $M(\lambda)$ has no purely imaginary eigenvalues. Denote the region of the complex plane for which this property is true by Ω, and assume that $\{\lambda : \mathrm{Re}\,\lambda > 0\} \subset \Omega$. This assumption implies that any temporal exponential instability will be due solely to the presence of point spectrum. Assume that for $\lambda \in \Omega$ that $M(\lambda)$ has m eigenvalues with positive real part, say $\mu_i^{+}(\lambda)$ for $i = 1, \ldots, m$, and $n - m$ eigenvalues with negative real part, say $\mu_i^{-}(\lambda)$ for $i = 1, \ldots, n - m$. The eigenvectors associated with $\mu_i^{\pm}(\lambda)$ will be denoted by $\eta_i^{\pm}(\lambda)$.

We will now construct the Evans function using various formulations. The central idea in all cases will be that we wish to create an analytic function which vanishes precisely when there is a localized solution to the eigenvalue problem. This requires that we construct solutions to equation (3.1) which decay as $z \to \pm\infty$.

3.1. Construction with simple eigenvalues

Let us first assume that the eigenvalues $\mu_i^{\pm}(\lambda)$ are simple. The eigenvectors can then be chosen to be analytic (Kato [27]), and solutions $\mathbf{Y}_i^{\pm}(\lambda, z)$ to equation (3.1), analytic in λ for fixed z, can be constructed so that

$$\lim_{z\to-\infty} \mathbf{Y}_i^{-}(\lambda, z)\mathrm{e}^{-\mu_i^{+}(\lambda)z} = \eta_i^{+}(\lambda), \quad i = 1, \ldots, m, \tag{3.2}$$

and

$$\lim_{z\to+\infty} \mathbf{Y}_i^+(\lambda,z)\mathrm{e}^{-\mu_i^-(\lambda)z} = \eta_i^-(\lambda), \quad i=1,\dots,n-m \tag{3.3}$$

([3]). The Evans function, $E(\lambda)$, is given by the scaled Wronskian of these solutions, i.e.,

$$E(\lambda) = m(\lambda,z)\det(\mathbf{Y}_1^-,\cdots,\mathbf{Y}_m^-,\mathbf{Y}_1^+,\cdots,\mathbf{Y}_{n-m}^+)(\lambda,z), \tag{3.4}$$

where

$$m(\lambda,z) = \exp\left(-\int_0^x \operatorname{trace}(M(\lambda)+R(s))\,\mathrm{d}s\right).$$

As a consequence of Abel's formula, the Evans function is independent of z. In the above context, the following theorem was proved by Alexander et al. [3].

Theorem 3.1. *The Evans function is analytic on Ω. Furthermore, $E(\lambda)=0$ if and only if λ is an eigenvalue, and the order of the zero is equal to the algebraic multiplicity of the eigenvalue.*

3.2. Construction via inner products

Under a slight relaxation of the above criteria, i.e., if one assumes only that the eigenvalues $\mu_i^-(\lambda)$ are simple, then there is an equivalent formulation of the Evans function, which is due to Swinton [39]. The equivalence was shown by Bridges and Derks [8]; furthermore, this new formulation has been exploited by Bridges [7], Bridges and Derks [9, 10] for a series of problems in which the eigenvalue problem has a Hamiltonian formulation. Consider the adjoint system associated with equation (3.1),

$$\mathbf{Z}' = -(M(\lambda)^* + R(z))^{\mathrm{T}}\mathbf{Z}. \tag{3.5}$$

The eigenvalues of $(-M(\lambda)^*)^{\mathrm{T}}$ are given by $-(\mu_i^\pm(\lambda))^*$; let the associated eigenvectors be given by $\zeta_i^\pm(\lambda)$. As above, one can construct solutions $\mathbf{Z}_i^+(\lambda,z)$ to equation (3.5) which satisfy

$$\lim_{z\to+\infty} \mathbf{Z}_i^+(\lambda,z)\mathrm{e}^{\mu_i^+(\lambda)^* z} = \zeta_i^+(\lambda), \quad i=1,\dots,m. \tag{3.6}$$

The Evans matrix is the generalization of the formulation of equation (2.13), and is given by

$$\mathbb{D}(\lambda) = \begin{pmatrix} \langle \mathbf{Y}_1^-, \mathbf{Z}_1^+\rangle & \cdots & \langle \mathbf{Y}_1^-, \mathbf{Z}_m^+\rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{Y}_m^-, \mathbf{Z}_1^+\rangle & \cdots & \langle \mathbf{Y}_m^-, \mathbf{Z}_m^+\rangle \end{pmatrix}(\lambda,z), \quad \lambda\in\Omega, \tag{3.7}$$

where now $\langle\cdot,\cdot\rangle$ represents the inner product on $\mathbb{C}^n$.

Theorem 3.2. *Set $D(\lambda)=\det(\mathbb{D}(\lambda))$. There exists an analytic function $C(\lambda)\neq 0$ such that $E(\lambda) = C(\lambda)D(\lambda)$.*

Remark 3.3. Without loss of generality it can henceforth be assumed that $C(\lambda)=1$.

3.3. Construction via exponential dichotomies

Unfortunately, it turns out that in many problems of interest the assumption that the eigenvalues of $M(\lambda)$ are simple for $\lambda\in\Omega$ does not hold. As a consequence, in order to preserve the analyticity of the Evans function, solutions to equation (3.1) must be constructed in a different manner [38]. Denote by $\Phi(\lambda;z,y)$ the evolution associated with equation (3.1). As discussed in [22, 38] (also see [11, 36] and the references therein) one can construct projection operators, $P_s(\lambda)$ and $P_u(\lambda)$, analytic in $\lambda\in\Omega$, such that for some $\kappa_s<0<\kappa_u$ and $K\geq 1$,

$$|\Phi(\lambda;z,0)P_u(\lambda)| \leq K\mathrm{e}^{\kappa_u z},\ z\leq 0; \quad |\Phi(\lambda;z,0)P_s(\lambda)| \leq K\mathrm{e}^{\kappa_s z},\ z\geq 0.$$

Furthermore, these operators have the property that

$$\dim \mathrm{R}(P_u(\lambda)) = m, \quad \dim \mathrm{R}(P_s(\lambda)) = n - m;$$

thus, they are maximal in the sense that they capture all of the initial data leading to exponentially decaying solutions as $z \to \pm\infty$. Given a $\mathbf{Y}_0 \in \mathbb{C}^n$, set $\mathbf{Y}^{\pm}(\lambda, z)$ to be

$$\mathbf{Y}^-(\lambda, z) = \Phi(\lambda; z, 0)P_u(\lambda)\mathbf{Y}_0, \quad \mathbf{Y}^+(\lambda, z) = \Phi(\lambda; z, 0)P_s(\lambda)\mathbf{Y}_0.$$

Note that if the eigenvalues of $M(\lambda)$ are simple, then

$$\mathbf{Y}^-(\lambda, z) \in \mathrm{Span}\{\mathbf{Y}_1^-(\lambda, z), \ldots, \mathbf{Y}_m^-(\lambda, z)\},$$

and

$$\mathbf{Y}^+(\lambda, z) \in \mathrm{Span}\{\mathbf{Y}_1^+(\lambda, z), \ldots, \mathbf{Y}_{n-m}^+(\lambda, z)\}.$$

It is clear that an initial condition will lead to a bounded solution if and only if $\mathbf{Y}^-(\lambda, 0) \cap \mathbf{Y}^+(\lambda, 0) \neq \{\mathbf{0}\}$.

Since the projections are analytic, as a consequence of [27, Chapter II.4.2] one can choose analytic bases $\{b_1(\lambda), \ldots, b_m(\lambda)\}$ and $\{b_{m+1}(\lambda), \ldots, b_n(\lambda)\}$ of $\mathrm{R}(P_u(\lambda))$ and $\mathrm{R}(P_s(\lambda))$, respectively. If one defines the analytic matrix $B(\lambda) \in \mathbb{C}^{n\times n}$ via

$$B(\lambda) = (b_1(\lambda) \cdots b_m(\lambda)\, b_{m+1}(\lambda) \cdots b_n(\lambda)),$$

then by construction it is clear that for $\lambda \in \Omega$ a uniformly bounded solution to equation (3.1) will exist if and only if $\dim \mathrm{N}(B(\lambda)) \geq 1$. The Evans function can then be defined to be

$$E(\lambda) = \det(B(\lambda)). \tag{3.8}$$

4. The linearization of the nonlinear Schrödinger equation

Let us now apply the results of the previous section to a well-understood example. The nonlinear Schrödinger hierarchy is the class of nonlinear integrable Hamiltonian systems given by

$$\mathbf{u}_t = \mathcal{K}(\mathbf{u}), \tag{4.1}$$

where $\mathbf{u} = (r, q) \in L^2(\mathbb{R}, \mathbb{C}^2)$,

$$\mathcal{K}(\mathbf{u}) = -2\sigma_3\Omega(\mathcal{L}^A(\mathbf{u}))\mathbf{u},$$

σ_3 is the Pauli spin matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\mathcal{L}^A$ is the integro-differential operator

$$\mathcal{L}^A(\mathbf{u})\mathbf{v} = -\frac{\mathrm{i}}{2}\partial_x\mathbf{v} + \mathrm{i}\mathbf{u}\int_{-\infty}^{x} [qv_1 - rv_2]\,\mathrm{d}y,$$

and $\Omega(\cdot) = \mathrm{i}P(\cdot)$, where $P(\cdot)$ is a polynomial with real-valued coefficients. The notation $\mathbf{v} = (v_1, v_2)$ is being used. If $P(k) = 1/2 + k^2$, then the evolution equation (4.1) is

$$\begin{aligned} q_t &= \mathrm{i}(\frac{1}{2}q_{xx} - q - q^2r) \\ r_t &= -\mathrm{i}(\frac{1}{2}r_{xx} - r - qr^2), \end{aligned} \tag{4.2}$$

which is the focusing nonlinear Schrödinger equation (NLS) upon setting $r = -q^*$. The interested reader is referred to [1, 2] for further details.

Assume that $\mathbf{u}_0$ is a stationary 1-soliton solution of equation (4.1). The interest is in the spectrum of the linearization $\mathcal{K}'(\mathbf{u}_0)$ about $\mathbf{u}_0$ and, in particular, in eigenvalues λ for which

$$\mathcal{K}'(\mathbf{u}_0)\mathbf{u} = \lambda\mathbf{u}$$

has a non-zero solution $\mathbf{u}$ in $L^2(\mathbb{R}, \mathbb{C}^2)$. Note that the eigenvalue problem for the focusing NLS equation (4.2) reads

$$\begin{aligned} \lambda q &= \mathrm{i}(\frac{1}{2}q_{xx} - q - 2q_0r_0q - q_0^2r) \\ \lambda r &= -\mathrm{i}(\frac{1}{2}r_{xx} - r - 2q_0r_0r - r_0^2q) \end{aligned} \tag{4.3}$$

with $\mathbf{u}_0 = (q_0, r_0)$. This problem can be solved if we can construct, and compute, the Evans function associated with the operator $\mathcal{K}'(\mathbf{u}_0)$. The essential spectrum of $\mathcal{K}'(\mathbf{u}_0)$ is given by

$$\sigma_{\mathrm{e}}(\mathcal{K}'(\mathbf{u}_0)) := \{\lambda \in \mathbb{C};\ \lambda = \pm 2\Omega(k),\ k \in \mathbb{R}\} \subset \mathrm{i}\mathbb{R}.$$

Also, it was shown in [25] that $\lambda = 0$ is the only eigenvalue of $\mathcal{K}'(\mathbf{u}_0)$. These eigenvalues at $\lambda = 0$ are due to the invariances associated with equation (4.1). We will recover this result later in this section for the polynomial dispersion relation P that gives the nonlinear Schrödinger equation.

The key to calculating the Evans function is to exploit Inverse Scattering Theory, which is possible since equation (4.1) is integrable. The underlying linear scattering problem associated with the nonlinear operator $\mathcal{K}(\mathbf{u})$ is the Zakharov-Shabat problem [2]

$$\mathbf{v}_x = \begin{pmatrix} -\mathrm{i}k & q_0(x) \\ -q_0^*(x) & \mathrm{i}k \end{pmatrix} \mathbf{v}$$

where $k \in \mathbb{C}$ is a complex parameter. The Jost functions are solutions to the Zakharov-Shabat eigenvalue problem that satisfy certain boundary conditions at $x = \pm\infty$. Appropriate quadratic combinations of the Jost functions define the adjoint squared eigenfunctions, which we denote[1] by $\Psi^{\mathrm{A}}(k, x)$, defined for $\operatorname{Im} k \geq 0$, and $\bar{\Psi}^{\mathrm{A}}(k, x)$, defined for $\operatorname{Im} k \leq 0$. The adjoint squared eigenfunctions are crucial ingredients when applying Soliton Perturbation Theory [29, 30]. As we shall see below, they can also be used to explicitly calculate the Evans function associated with $\mathcal{K}'(\mathbf{u}_0)$. For $k \in \mathbb{R}$, the adjoint squared eigenfunctions satisfy the identities

$$[\mathcal{L}^{\mathrm{A}}(\mathbf{u}) - k]\Psi^{\mathrm{A}}(k, x) = [\mathcal{L}^{\mathrm{A}}(\mathbf{u}) - k]\bar{\Psi}^{\mathrm{A}}(k, x) = 0.$$

Furthermore, they have the property that, for fixed x, $\Psi^{\mathrm{A}}(k, x)$ is analytic in k for $\operatorname{Im} k > 0$, while $\bar{\Psi}^{\mathrm{A}}(k, x)$ is analytic in k for $\operatorname{Im} k < 0$. In addition, they have the asymptotics

$$\lim_{x\to-\infty} \Psi^{\mathrm{A}}(k, x)\mathrm{e}^{2\mathrm{i}kx} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \lim_{x\to\infty} \Psi^{\mathrm{A}}(k, x)\mathrm{e}^{2\mathrm{i}kx} = a(k)^2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{4.4}$$

and

$$\lim_{x\to-\infty} \bar{\Psi}^{\mathrm{A}}(k, x)\mathrm{e}^{-2\mathrm{i}kx} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lim_{x\to\infty} \bar{\Psi}^{\mathrm{A}}(k, x)\mathrm{e}^{-2\mathrm{i}kx} = \bar{a}(k)^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.5}$$

The functions $a(k)$ and $\bar{a}(k)$ are the transmission coefficients for the Zakharov-Shabat eigenvalue problem, and for the 1-soliton of the NLS equation (4.2) are given by

$$a(k) = \frac{\sqrt{2}\,k - \mathrm{i}}{\sqrt{2}\,k + \mathrm{i}}, \quad \bar{a}(k) = \frac{\sqrt{2}\,k + \mathrm{i}}{\sqrt{2}\,k - \mathrm{i}} \tag{4.6}$$

[28, 30]. Furthermore, it is known, see [25] for references, that the adjoint squared eigenfunctions and the transmission coefficients can be extended analytically across the line $\operatorname{Im} k = 0$.

[1] Notation: We denote by q^* the complex conjugate of a complex number q. Thus, $\bar{q}$ does *not* refer to the complex conjugate.

Lemma 4.1 ([25]). *The adjoint squared eigenfunctions satisfy*

$$[\mathcal{K}'(\mathbf{u}_0) - 2\Omega(k)]\sigma_3\Psi^{\mathrm{A}}(k,x) = 0, \qquad \operatorname{Im} k \geq 0$$
$$[\mathcal{K}'(\mathbf{u}_0) + 2\Omega(\bar{k})]\sigma_3\bar{\Psi}^{\mathrm{A}}(\bar{k},x) = 0, \qquad \operatorname{Im}\bar{k} \leq 0.$$

Remark 4.2. The interested reader should also consult [40, 41].

Thus, upon applying the matrix σ_3 to the squared eigenfunctions, one recovers eigenfunctions for the linearized problem. We are now in the position to calculate the Evans function. While it is possible to make this calculation for the full hierarchy (see [25] for the details), for the sake of clarity only the specific dispersion relation

$$\Omega(\ell) = \mathrm{i}\left(\frac{1}{2} + \ell^2\right) \tag{4.7}$$

of the nonlinear Schrödinger equation will be considered. For this case explicit expressions are known [30] for the adjoint squared eigenfunctions given in Lemma 4.1; however, as it will be seen below, these expressions are not necessary in order to make a calculation.

Consider equation (4.7) and note that the associated continuous spectrum consists of the elements $\lambda = \pm\mathrm{i}(1+2\ell^2)$ where ℓ varies in $\mathbb{R}$, i.e., it is the imaginary axis minus the interval $(-\mathrm{i},\mathrm{i})$. Since we wish to exploit Lemma 4.1, we choose for each $\lambda \notin \sigma_{\mathrm{e}}(\mathcal{K}'(\mathbf{u}_0))$ numbers k and $\bar{k}$ such that

$$\lambda = 2\Omega(k) = \mathrm{i}(1+2k^2), \quad \lambda = -2\Omega(\bar{k}) = -\mathrm{i}(1+2\bar{k}^2),$$

i.e.,

$$k(\lambda) = \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}3\pi/4}\sqrt{\lambda-\mathrm{i}}, \quad \arg(\lambda-\mathrm{i}) \in (-3\pi/2, \pi/2], \tag{4.8}$$

and

$$\bar{k}(\lambda) = \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}3\pi/4}\sqrt{\lambda+\mathrm{i}}, \quad \arg(\lambda-\mathrm{i}) \in (-\pi/2, 3\pi/2], \tag{4.9}$$

The branch cuts have been chosen so that $\operatorname{Im} k > 0$ and $\operatorname{Im}\bar{k} < 0$, and that k and $\bar{k}$ are analytic for $\lambda \notin \sigma_{\mathrm{e}}(\mathcal{K}'(\mathbf{u}_0))$. By Lemma 4.1 it is seen that $\sigma_3\Psi^{\mathrm{A}}(k,x)$ and $\sigma_3\bar{\Psi}^{\mathrm{A}}(\bar{k},x)$ are two solutions to the eigenvalue problem equation (4.3). Both of these solutions decay exponentially fast to zero as $x \to -\infty$. Furthermore, as a consequence of equation (4.4) and equation (4.5), we know their asymptotics as $x \to \infty$. From now on, we regard k and $\bar{k}$ as functions of λ defined via equation (4.8) and equation (4.9).

Next, rewrite the eigenvalue problem equation (4.3) as an ODE

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = [M(\lambda) + R(x)]\mathbf{Y}, \tag{4.10}$$

where $\mathbf{Y} = (q, r, q_x, r_x)^{\mathrm{T}} \in \mathbb{C}^4$, and

$$M(\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2\mathrm{i}(\lambda+\mathrm{i}) & 0 & 0 & 0 \\ 0 & 2\mathrm{i}(\lambda-\mathrm{i}) & 0 & 0 \end{pmatrix}, \qquad R(x) = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2q_0r_0 & q_0^2 & 0 & 0 \\ r_0^2 & 2q_0r_0 & 0 & 0 \end{pmatrix}.$$

Note that $|R(x)| \to 0$ as $|x| \to \infty$. The eigenvalues and eigenvectors of $M(\lambda)$ are analytic for $\lambda \notin \sigma_{\mathrm{e}}(\mathcal{K}'(\mathbf{u}_0))$. We can define two solutions[2]

$$\mathbf{Y}_1^{\mathrm{u}}(\lambda,x) = \begin{pmatrix} 1 \\ \partial_x \end{pmatrix}\sigma_3\Psi^{\mathrm{A}}(k(\lambda),x), \qquad \mathbf{Y}_2^{\mathrm{u}}(\lambda,x) = \begin{pmatrix} 1 \\ \partial_x \end{pmatrix}\sigma_3\bar{\Psi}^{\mathrm{A}}(\bar{k}(\lambda),x)$$

that decay exponentially to zero as $x \to -\infty$. Analogously, there are two solutions to the adjoint equation associated with equation (4.10) which decay exponentially as $x \to +\infty$ for $\lambda \notin \sigma_{\mathrm{e}}(\mathcal{K}'(\mathbf{u}_0))$:

$$\begin{aligned} \lim_{x\to+\infty}\mathbf{Z}_1^{\mathrm{s}}(\lambda,x)\mathrm{e}^{2\mathrm{i}k(\lambda)^*x} &= (0, 2\mathrm{i}k(\lambda)^*, 0, 1)^{\mathrm{T}} \\ \lim_{x\to+\infty}\mathbf{Z}_2^{\mathrm{s}}(\lambda,x)\mathrm{e}^{-2\mathrm{i}\bar{k}(\lambda)^*x} &= (-2\mathrm{i}\bar{k}(\lambda)^*, 0, 1, 0)^{\mathrm{T}}. \end{aligned} \tag{4.11}$$

[2]Note that the subscripts of $\mathbf{Y}_j^{\mathrm{u}}$ do *not* denote the components of the vectors.

Following the formulation of Section 3.2 the Evans matrix is given by

$$\mathbb{D}(\lambda) = \begin{pmatrix} \langle \mathbf{Y}_1^{\mathrm{u}}, \mathbf{Z}_1^{\mathrm{s}} \rangle & \langle \mathbf{Y}_1^{\mathrm{u}}, \mathbf{Z}_2^{\mathrm{s}} \rangle \\ \langle \mathbf{Y}_2^{\mathrm{u}}, \mathbf{Z}_1^{\mathrm{s}} \rangle & \langle \mathbf{Y}_2^{\mathrm{u}}, \mathbf{Z}_2^{\mathrm{s}} \rangle \end{pmatrix} (\lambda, x), \quad \lambda \notin \sigma_{\mathrm{e}}(\mathcal{K}'(\mathbf{u}_0)), \tag{4.12}$$

and the Evans function is given by $E(\lambda) = \det(\mathbb{D}(\lambda))$. Now we evaluate the Evans function. As a consequence of equation (4.4) and equation (4.5), we have the asymptotics

$$\begin{aligned} \lim_{x \to +\infty} \mathbf{Y}_1^{\mathrm{u}}(\lambda, x) \mathrm{e}^{2\mathrm{i}k(\lambda)x} &= a(k(\lambda))^2 \, (0, 1, 0, -2\mathrm{i}k(\lambda))^{\mathrm{T}} \\ \lim_{x \to +\infty} \mathbf{Y}_2^{\mathrm{u}}(\lambda, x) \mathrm{e}^{-2\mathrm{i}\bar{k}(\lambda)x} &= \bar{a}(\bar{k}(\lambda))^2 \, (1, 0, 2\mathrm{i}\bar{k}(\lambda), 0)^{\mathrm{T}}, \end{aligned} \tag{4.13}$$

where

$$a(k(\lambda)) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\lambda - \mathrm{i}} - 1}{\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\lambda - \mathrm{i}} + 1}, \quad \bar{a}(\bar{k}(\lambda)) = \frac{\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{\lambda + \mathrm{i}} - 1}{\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{\lambda + \mathrm{i}} + 1}.$$

Taking the limit $x \to \infty$ in equation (4.12) and using the asymptotics given in equation (4.11) and equation (4.13) yields

$$E(\lambda) = 8a(k(\lambda))^2 \bar{a}(\bar{k}(\lambda))^2 \sqrt{\lambda - \mathrm{i}} \sqrt{\lambda + \mathrm{i}}. \tag{4.14}$$

As a final remark, note that $a(k(0)) = \bar{a}(\bar{k}(0)) = 0$, and that these zeros are simple.

4.1. Edge bifurcations

For the rest of the discussion in this section, assume that $\operatorname{Im} \lambda \geq 0$. An examination of the Evans function reveals that $\lambda = 0$ is a zero of order four, and that the only other zeros are at the branch points $\lambda = \pm\mathrm{i}$. The eigenvalues at the origin are due to the symmetries associated with the NLS. Now consider $\lambda = \mathrm{i}$. It is clear that $\sigma_3 \Psi^{\mathrm{A}}(k(\mathrm{i}), x)$ is uniformly bounded but nondecaying for all x, while $\sigma_3 \bar{\Psi}^{\mathrm{A}}(\bar{k}(\mathrm{i}), x)$ decays exponentially as $x \to -\infty$ and grows exponentially fast as $x \to +\infty$. Hence, $\lambda = \mathrm{i}$ is not a true eigenvalue, as there exist no corresponding eigenfunctions which are localized in space.

What is the effect of this spurious zero? Set

$$\gamma^2 := \lambda - \mathrm{i}. \tag{4.15}$$

This transformation defines a Riemann surface, and with respect to the Evans function the principal sheet is given by $\arg(\gamma) \in (-3\pi/4, \pi/4]$. Zeros of the Evans function on this sheet correspond to true eigenvalues, whereas zeros on the other sheet correspond to resonance poles. On the Riemann surface the Evans function is given by

$$E(\gamma) = 8a(k(\gamma))^2 \bar{a}(\bar{k}(\gamma))^2 \gamma \sqrt{\gamma^2 + 2\mathrm{i}}.$$

The Evans function is analytic on the Riemann surface, and in addition to the four zeros at $\gamma = \mathrm{e}^{-\mathrm{i}\pi/4}$ ($\lambda = 0$), it has a simple zero at $\gamma = 0$.

Now suppose that the NLS undergoes a smooth perturbation of $\mathrm{O}(\epsilon)$. Assuming that the Evans function remains analytic on the Riemann surface, the zero at $\gamma = 0$ will generically move onto one of the two sheets and also be of $\mathrm{O}(\epsilon)$. If the perturbed zero is on the principal sheet, then an eigenvalue has been created via an edge bifurcation, and from equation (4.15) it is seen that it will be $\mathrm{O}(\epsilon^2)$ from the branch point $\lambda = \mathrm{i}$. Thus, upon perturbation an eigenvalue can be created where once there was none. The spurious zero then leads to the potential creation of an eigenvalue upon a perturbation of the vector field. This is illustrated in Figure 1.

While this is an interesting topic, it is beyond the scope of this article. The interested reader is referred to the review article [**26**] and the references therein. For a full treatment of the NLS hierarchy one should consult [**25**].

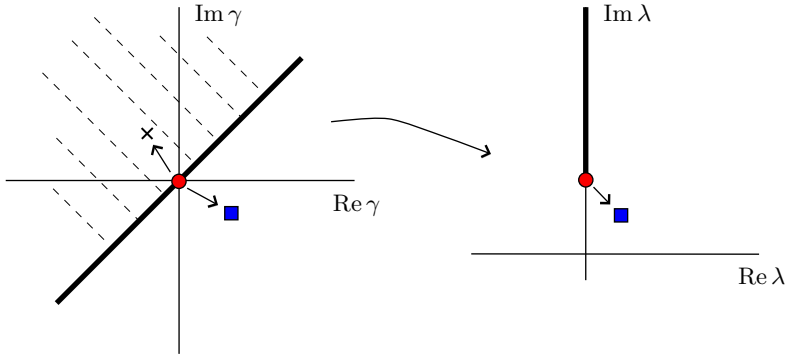

Figure 1: The Riemann surface is given in the left panel, and the spectral plane is in the right panel. The physical sheet of the Riemann surface satisfies $\arg(\gamma) \in (-3\pi/4, \pi/4]$, and the boundary between the two sheets is given by $\mathrm{Re}\,\gamma = \mathrm{Im}\,\gamma$. The circle represents the zero of the Evans function for the unperturbed problem, and the squares and crosses represent the possible movement of the zero under perturbation. Note that an eigenvalue is created only if the zero on the Riemann surface moves onto the physical sheet.

5. Dissipative Perturbations

In the previous sections the spectrum was located via an Evans function calculation. The calculation was performed on a class of well-understood problems. In this section we will start with the assumption that the unperturbed problem is well-understood, and then use the Evans function to understand the perturbed problem. In particular, we are interested in dissipative perturbations of Hamiltonian systems.

5.1. Theoretical results

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, let J is an invertible skew-symmetric operator with bounded inverse, and consider the Hamiltonian system on H given by

$$\frac{\mathrm{d}v}{\mathrm{d}t} = JE'(v)$$

which respects a finite-dimensional abelian connected Lie group $\mathscr{G}$ with Lie algebra $\mathfrak{g}$ on H. Assume that $\dim(\mathfrak{g}) = n$. It will be assumed that the system is invariant under the action of a unitary representation T. We seek relative equilibria of the form

$$v(t) = T(\exp(\omega t))v_0,$$

for appropriate $\omega \in \mathfrak{g}$. Therefore, change variables and consider

$$v(t) = T(\exp(\omega t))u(t),$$

so that u satisfies

$$\frac{\mathrm{d}u}{\mathrm{d}t} = JE_0'(u;\omega), \tag{5.1}$$

where

$$E_0'(u;\omega) := E'(u) - J^{-1}T_\omega u.$$

Here T_ω is the skew-symmetric operator which is the generator of $\exp(\omega t)$. The new Hamiltonian depends therefore on ω. We assume that the steady-state equation

$$E_0'(u;\omega) = 0$$

has a smooth family $\Phi(\omega)$ of solutions, where ω varies in $\mathfrak{g}$.

Let the linear operator about the wave Φ be denoted by JE_0''. Since $\mathscr{G}$ is abelian, it is known that the operator JE_0'' will have a nontrivial kernel:

$$JE_0''(\Phi)T_{\omega_i}\Phi = 0, \quad JE_0''(\Phi)\partial_{\omega_i}\Phi = T_{\omega_i}\Phi, \tag{5.2}$$

for $i = 1, \ldots, n$, with the set $\{T_{\omega_1}\Phi, \ldots, T_{\omega_n}\Phi\}$ being orthogonal. Here $\partial_\sigma := \partial/\partial\sigma$ for $\sigma \in \mathfrak{g}$. Furthermore, if the symmetric matrix $D_0 \in \mathbb{R}^{n\times n}$ given by

$$(D_0)_{ij} := \langle \partial_{\omega_j}\Phi, E_0''\partial_{\omega_i}\Phi\rangle \tag{5.3}$$

is invertible, then this set is a basis for the kernel. Thus, under the assumption that D_0 is nonsingular, when considering the eigenvalue problem $JE_0''u = \lambda u$ one has that $\lambda = 0$ is an eigenvalue with geometric multiplicity n and algebraic multiplicity $2n$.

Now consider the perturbed problem given by

$$\frac{\mathrm{d}u}{\mathrm{d}t} = JE_0'(u) + \epsilon E_1(u), \tag{5.4}$$

where E_1 represents the dissipative perturbation. It will be assumed that while the perturbation breaks the Hamiltonian structure, it does not remove any of the symmetries. When considering the persistence of the wave, a standard Lyapunov-Schmidt reduction reveals the following (see [21] for the case of Hamiltonian perturbations):

Lemma 5.1. *The wave $u = \Phi(\omega) + \epsilon\Phi_\epsilon + \mathrm{O}(\epsilon^2)$ will persist only if the condition*

$$\langle J^{-1}E_1(\Phi(\omega)), T_{\omega_j}\Phi\rangle = 0, \quad j = 1, \ldots, n \tag{5.5}$$

holds for some $\omega \in \mathfrak{g}$. Furthermore,

$$\Phi_\epsilon = -(E_0''(\Phi(\omega))J^{-1}E_1(\Phi(\omega)).$$

The above lemma yields a necessary, but not sufficient, condition for persistence. Since the perturbation does not destroy any of the symmetries associated with the original system, the eigenfunction $T_{\omega_j}\Phi$ will persist; in particular, one has the formal expansion $T_{\omega_j}\Phi + \epsilon T_{\omega_j}\Phi_\epsilon + \mathrm{O}(\epsilon^2)$. Now write the linearization about the perturbed wave as $JE_0'' + \epsilon L_\epsilon + \mathrm{O}(\epsilon^2)$, and define the matrix $M \in \mathbb{R}^{n\times n}$ by

$$M_{ij} := \langle L_\epsilon\partial_{\omega_j}\Phi - T_{\omega_j}\Phi_\epsilon, J^{-1}T_{\omega_i}\Phi\rangle.$$

Lemma 5.2. *Suppose that M is nonsingular. Then equation (5.5) is also sufficient.*

Since the Hamiltonian structure has been broken, n eigenvalues of $\mathrm{O}(\epsilon)$ will leave the origin. Another Lyapunov-Schmidt reduction, similar to that in [21, Theorem 4.4] (also see the more general case in [22, Section 4.3]), allows one to track the location of these eigenvalues.

Lemma 5.3. *The nonzero $\mathrm{O}(\epsilon)$ eigenvalues and associated eigenfunctions for the perturbed eigenvalue problem*

$$(JE_0'' + \epsilon L_\epsilon)u = \lambda u$$

are given by

$$\lambda = \epsilon\lambda_1 + \mathrm{O}(\epsilon^2), \quad u = \sum_{j=1}^{n} v_j T_{\omega_j}\Phi + \mathrm{O}(\epsilon),$$

where λ_1 is the eigenvalue and $\mathbf{v}$ is the associated eigenvector for the generalized eigenvalue problem

$$(D_0\lambda_1 - M)\mathbf{v} = \mathbf{0}.$$

Remark 5.4. In [20] it was shown in specific examples that the perturbation expansion for the Evans function satisfies

$$E(\lambda, \epsilon) = \det(D_0\lambda - M\epsilon) + \mathrm{O}(\epsilon^2).$$

In general, however, it is easier to find the eigenvalues of a matrix rather than locate the zeros of its characteristic polynomial; hence, the formulation in Lemma 5.3.

Now, it may also be possible for eigenvalues to pop out of the essential spectrum, creating internal modes via an edge bifurcation [24, 25, 31, 35] (also see Section 4.1). Since these eigenvalues will be of $\mathrm{O}(1)$, they will not be captured by the perturbation expansion given in Lemma 5.3. However, as it will be seen in the example in the next section, this is not necessarily problematic.

5.2. Example: Nonlinear Schrödinger equation

The theoretical results will now be applied to a particular dissipative perturbation of the nonlinear Schrödinger equation. Consider

$$\mathrm{i}\partial_t q + \frac{1}{2}\partial_x^2 q - \omega q + |q|^2 q = \mathrm{i}\epsilon R(q, q^*), \tag{5.6}$$

where

$$R(q, q^*) := \frac{1}{2} d_1 \partial_x^2 q + d_2 q + d_3 |q|^2 q + d_4 |q|^4 q.$$

Here $d_j \in \mathbb{R}$ and $\omega \in \mathbb{R}^+$. Here $d_1 > 0$ describes spectral filtering, and a consequence of the sign is that the perturbed equation is dissipative. The parameter $d_2 < 0$ accounts for linear loss in the fiber, and a consequence of this sign is that any instability of a wave arises only from point spectrum. The parameter d_3 accounts for nonlinear gain or loss, and d_4 represents a higher-order correction to the nonlinear gain or loss.

The unperturbed problem is well-understood (see Section 4). Solutions to equation (5.6) are invariant under the action

$$T(\xi, \theta) q = q(x + \xi) \mathrm{e}^{i\theta},$$

so that the unperturbed problem has two symmetries. The linearized problem associated with the unperturbed problem is as follows. After writing equation (5.6) in real and imaginary parts, it is seen that the linearization when $\epsilon = 0$ is JL_0, where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L_0 = \mathrm{diag}(L_r, L_i),$$

with

$$L_r := -\frac{1}{2}\partial_x^2 + \omega - 3Q_0(x)^2, \quad L_i := -\frac{1}{2}\partial_x^2 + \omega - Q_0(x)^2.$$

Here, the soliton is given by $Q_0(x) := \sqrt{2\omega}\,\mathrm{sech}(\sqrt{2\omega}\,x)$. Furthermore, one has that the eigenfunctions given in equation (5.2) are

$$T_\xi Q_0 = \partial_x Q_0\, \mathbf{e}_1, \quad \partial_\xi Q_0 = -x Q_0\, \mathbf{e}_2,$$

and

$$T_\phi Q_0 = Q_0\, \mathbf{e}_2, \quad \partial_\phi Q_0 = \partial_\omega Q_0\, \mathbf{e}_1.$$

Here $\mathbf{e}_j \in \mathbb{R}^2$ represents the j^{th} unit vector.

Lemma 5.1 states a necessary condition for the persistence of the wave Q_0. A routine calculation shows that the condition reduces to

$$\langle R(Q_0, Q_0^*), Q_0 \rangle = 0,$$

i.e.,

$$e(\omega) := d_4 \omega^2 + \frac{5}{32}(4d_3 - d_1)\omega + \frac{15}{32} d_2 = 0.$$

Define

$$d_4^* := \frac{5}{384} \frac{(4d_3 - d_1)^2}{d_2}.$$

Assuming that

$$4d_3 - d_1 > 0, \quad d_4^* < d_4 < 0,$$

$e(\omega) = 0$ has the solutions

$$\omega = \omega^\pm := -\frac{5}{64 d_4} \left((4d_3 - d_1) \pm \sqrt{(4d_3 - d_1)^2 - \frac{384}{5} d_2 d_4} \right).$$

For $\omega = \omega^\pm$ the perturbed wave is given by

$$q = \begin{pmatrix} Q_0(x) \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ Q_0(x) \int_0^x \theta(s)\, \mathrm{d}s \end{pmatrix} + \mathrm{O}(\epsilon^2), \tag{5.7}$$

where

$$\theta(s) = -\frac{1}{\sqrt{2\omega}} \tanh(\sqrt{2\omega}\, s) \left(d_1\omega + d_2 - \frac{8}{5} d_4 \omega^2 \operatorname{sech}^2(\sqrt{2\omega}\, s) \right).$$

Note that

$$\lim_{d_4 \to d_4^*} \omega^{\pm} = -6 \frac{d_2}{4d_3 - d_1}.$$

Thus, when considering the existence problem there is a saddle-node bifurcation at $d_4 = d_4^*$. One expects that at most only one of these perturbed waves will be stable. Also note that the condition $d_4 \neq 0$ is crucial for the bifurcation to occur. The calculations given below will show that if $d_4 = 0$, in which case

$$\omega = \lim_{d_4 \to 0} \omega^{-} = -3 \frac{d_2}{4d_3 - d_1},$$

then this wave will be unstable.

Now let us consider the stability of the perturbed wave. The matrix D_0 given in Lemma 5.3 is given by

$$\begin{aligned} D_0 &= \operatorname{diag}(\langle \Phi_x, -x\Phi \rangle, -\partial_\omega \langle \Phi, \Phi \rangle) \\ &= 2 \operatorname{diag}((2\omega)^{1/2}, -(2\omega)^{-1/2}), \end{aligned}$$

The first-order correction of the linearized operator is given by

$$L_\epsilon = (\frac{1}{2} d_1 \partial_x^2 + d_2)\,\mathrm{id} + d_3 Q_0^2 \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} + d_4 Q_0^4 \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} + 2Q_0^2 \int_0^x \theta(s)\, \mathrm{d}s \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Upon using the perturbation expansion in equation (5.7), the matrix M used in the stability calculation present in Lemma 5.3 is then given by

$$M(\omega^{\pm}) = \operatorname{diag}(C_1 d_1, \mp C_2 \sqrt{(4d_3 - d_1)^2 - 384 d_2 d_4 / 5}),$$

where C_i is a positive constant for $i = 1, 2$. Thus, upon using the result of Lemma 5.3 the location of the $\mathrm{O}(\epsilon)$ eigenvalues is given by

$$\begin{aligned} \lambda_{\mathrm{rot}} &= -C_{\mathrm{rot}} d_1 \epsilon + \mathrm{O}(\epsilon^2) \\ \lambda_{\mathrm{amp}} &= C_{\mathrm{amp}} \begin{cases} -\sqrt{(4d_3 - d_1)^2 - 384 d_2 d_4 / 5}\, \epsilon + \mathrm{O}(\epsilon^2), & \omega = \omega^+ \\ \sqrt{(4d_3 - d_1)^2 - 384 d_2 d_4 / 5}\, \epsilon + \mathrm{O}(\epsilon^2), & \omega = \omega^-. \end{cases} \end{aligned} \tag{5.8}$$

In the above the constants are again positive. The eigenvalue λ_{rot} is always negative, whereas the sign of the eigenvalue λ_{amp} depends upon the amplitude of the wave. It should be noted that the adiabatic approach yields the same conclusion as above [**32**, **37**]. The bifurcation diagram is given in Figure 2.

Unfortunately, the result of equation (5.8) is not enough to conclude that the perturbed wave with $\omega = \omega^+$ is stable, as all of the possible instability mechanisms have not yet been captured. As discussed in Section 4.1, it is possible for a single eigenvalue to leave the edge of the continuous spectrum upon perturbation. The location of this eigenvalue must now be determined.

The radiation modes themselves are readily computed. The upper branch is given by the dispersion relation

$$\lambda = \mathrm{i}(\omega + \frac{1}{2} k^2) + \epsilon(d_2 - \frac{1}{2} d_1) k^2, \quad k \in \mathbb{R}.$$

Since by assumption $d_2 < 0$ and $d_1 > 0$, the continuous spectrum moves into the left-half of the complex plane and does not act to destabilize the wave.

Now, as in Section 4.1, one must track the zero of the Evans function on the appropriate Riemann surface to determine if any eigenvalue arises from an edge bifurcation. As is seen in [**24**], the appropriate Riemann surface is defined by

$$\gamma^2 := (1 - \mathrm{i}\epsilon d_1)\lambda - \mathrm{i}\omega - \epsilon(d_1\omega + d_2) + \mathrm{i}\epsilon^2 d_1 d_2. \tag{5.9}$$

Since the zero γ_{edge} on the Riemann surface will satisfy $\gamma_{\mathrm{edge}} = \mathrm{O}(\epsilon)$, by solving the above for λ it is seen that if an eigenvalue pops out of the continuous spectrum due to an edge bifurcation, it is given by

$$\lambda_{\mathrm{edge}} = \mathrm{i}\omega + \epsilon d_2 + \mathrm{O}(\epsilon^2).$$

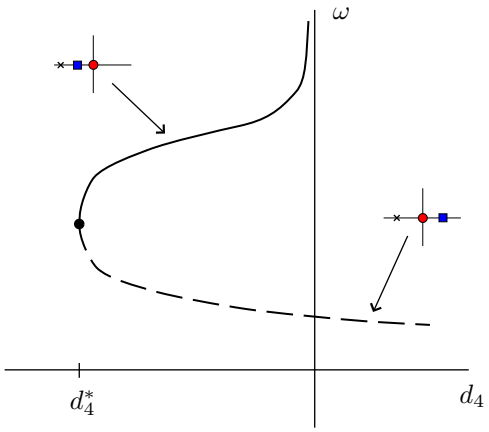

Figure 2: The set $e(\omega) = 0$ for $d_1 > 0$, $d_2 < 0$, and $d_3 > d_1/4$ fixed. The insets give the location of the $\mathrm{O}(\epsilon)$ eigenvalues for $\omega = \omega^+$ (solid line) and $\omega = \omega^-$ (dashed line). The cross represents λ_{rot} and the square represents λ_{amp}.

Thus, if $d_2 = \mathrm{O}(1)$, then it is necessarily true that λ_{edge} is contained in the left-half of the complex plane, and hence does not contribute to an instability.

However, now suppose that $d_2 = \mathrm{O}(\epsilon)$. It is then possible to have $\mathrm{Re}\,\lambda_{\mathrm{edge}} > 0$, which implies that it may be possible for the adiabatically stable wave with $\omega = \omega^+$ to destabilize via a Hopf bifurcation. In order to track this eigenvalue one needs to explicitly compute the Evans function on the Riemann surface given in equation (5.9), and then track the zero as it moves under the perturbation. This tedious calculation is carried out in [**23**]. It is determined therein that the zero moves onto the second sheet of the Riemann surface, and hence an eigenvalue is not created via an edge bifurcation. The perturbed wave with $\omega = \omega^+$ is then spectrally stable, and since the system is dissipative, this then implies that it is stable.

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